DIFFERENTIAL AND INTEGRAL CALCULUS FOR BEGINNERS.

DIFFERENTIAL

AND

INTEGRAL CALCULUS

FOR BEGINNERS

Adapted to the use of Students of Physics and Mechanics

BY

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London, Edinburgh, Dublin, and New York

1909

PREFACE.

IN commencing the study of advanced theoretical physics, students frequently find themselves hopelessly handicapped by their lack of knowledge of the higher mathematics. Most of the text-books on the Differential and Integral Calculus are unsuited to afford material assistance, being far too full and detailed in some respects, and deficient or entirely wanting in others. In order to assist students labouring under these difficulties, a series of articles was commenced in the pages of *The Practical Teacher* in the month of April 1898. In these it was sought to explain the essential principles of the Calculus shorn of all extraneous difficulties, and to lead the student by natural and easy stages to the application of these principles to important problems in mechanics and physics.

Letters received from time to time from a great variety of students showed that these articles met a real want, and it was ultimately decided to amplify and extend them, and publish them in book form. The present volume is the result of that decision.

In the following pages no attempt has been made to develop the subject as a branch of abstract mathematics; the aim has been to provide the physical student with a valuable engine of research, and to accustom him to its use. Many discussions found in larger treatises will, therefore, be unmentioned in the present work, as possessing no interest from a physical point of view. Lack of space must serve as an excuse for the omission of a few elementary investigations which possess a real value to the physical student.

The subject-matter has been arranged so that, though possessing no mathematical attainments further than a sound knowledge of elementary algebra and geometry, the student should find no difficulties which cannot be overcome by application and perseverance. The general nature of the most important functions is treated of in the opening chapter; an appendix gives a short synopsis of the trigonometrical formulæ subsequently used, with their proofs. By this means reference to other books is rendered unnecessary. The significance of a differential coefficient is then explained on geometrical grounds, and it is shown how a knowledge of the binomial theorem and a few trigonometrical formulæ can be used to obtain the differential coefficient of any function. Applications to maxima and minima problems, and the expansion of functions, are then given. Incidentally the student is introduced to the geometry of complex quantities.

The method of obtaining the integral of a function is explained on grounds more simple than those generally employed, and numerous applications to geometrical, mechanical, and physical problems are then given. In each case a short explanation of the nature of the problem in hand is provided, so that its physical aspect may be kept clearly in view throughout the investigation. Most of the problems chosen possess intrinsic importance; a few have been selected in order to show the use of particular mathematical devices.

The concluding chapter, on differential equations, though necessarily incomplete, should prove useful to the student as an introduction to a most important branch of mathematics. The increasing importance of hyperbolic functions in physical investigations renders any apology for introducing a short discussion of their properties unnecessary.

I am pleased to have this opportunity of returning my thanks to Mr. W. P. Workman, M.A., who has kindly read through the proofs of this book, for his helpful criticisms and suggestions.

DAVY-FARADAY LABORATORY,
ROYAL INSTITUTION,
October 1900.

CONTENTS.

DIFFERENTIAL AND INTEGRAL CALCULUS FOR BEGINNERS.

CHAPTER I.

INTRODUCTORY.

VARIABLES. — Quantities possessing variable values are generally represented by the English letters x, y, z, or the Greek letters ξ, η, ζ, etc. Quantities possessing constant values are generally represented by the English letters a, b, c, d, etc., and the Greek letters α, β, γ, δ, etc.

CARTESIAN CO-ORDINATES.—In order to define the position of a point in a plane, the following conventions are used. Let OX, OY (Fig. 1) be two fixed straight lines at right angles to each

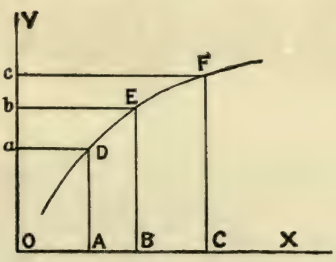

FIG. 1.

other, intersecting in the point O. These lines are termed the *axes of co-ordinates*, and the point O is termed the *origin*. Now the position of any point D becomes definitely known when DA and Da, its perpendicular distances from the axes, are given. If we write y for AD ($=$Oa), and x for OA ($=a$D), then we may say that the position of D is defined by the corresponding values of y and x. Thus y and x are termed the *co-ordinates* of the point D. Further, y is termed the *ordinate*, and x the *abscissa*, of the point D.*

In order to define the position of *any point* in a plane, let X′OX, Y′OY (Fig. 2) be the axes, intersecting at the origin O.

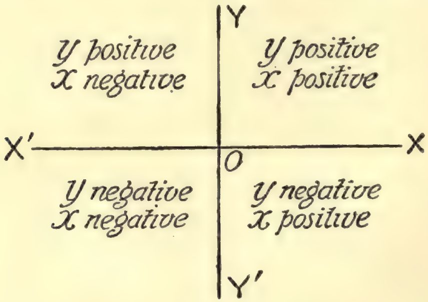

FIG. 2.

Wherever the point may be, perpendiculars may be drawn from it to the axes X′OX and Y′OY. If the corresponding value of x is measured *to the right from* O, its sign is positive ($+$); if *to the left from* O, its sign is negative ($-$). Values of y, measured upwards from X′OX, are positive, those measured downwards from X′OX being negative. Thus the axes X′OX and Y′OY (Fig. 2) divide the whole of the plane of the paper into four quadrants; and if the *signs* and *numerical magnitudes* of the co-ordinates of any point are given, the position of that point at once becomes known, according to the scheme indicated in Fig. 2.

The above system of co-ordinates is termed the *Cartesian* system, from Descartes, who first extensively developed it.

FUNCTIONS.—The expression $y=f(x)$—which reads y equals a function of x—is one that will very frequently be used. Its meaning is this: the value of y depends on the value which we give to another quantity, x. To make this clearer. Let OX, OY (Fig. 1) represent rectangular axes, and let a certain curve DEF be drawn.

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* Ordinate, from Lat. *ordino*, to set in a row, arrange. Abscissa (pl. abscissæ), from Lat. *abs*, away; *scido*, to cut. The abscissa is *cut off* from the axis of x by the ordinate y.

The position of any point on the curve is exactly defined when its perpendicular distance from each of the two axes of reference is given.

Now, it is obvious that the value of y will depend on the value of x which we choose. Thus to $x = \mathsf{OA}$ corresponds a value of $y = \mathsf{AD}$; to $x = \mathsf{OB}$ corresponds a value of $y = \mathsf{BE}$; to $x = \mathsf{OC}$ corresponds a value of $y = \mathsf{CF}$; and, generally, we may say that y depends for its value on the value of x chosen. This is analytically written $y = f(x)$. Instead of $f(x)$, any of the following expressions may be used :—$f(x)$, $\phi(x)$, $\psi(x)$, $\chi(x)$, etc. It must be carefully noticed that $f(x)$ is taken as a whole, and does not mean $f \times (x)$.

In the above diagram we may take various values for x, and to each one will correspond a definite value for y. x and y are therefore called variables; and since y depends for its value on the value chosen for x, y is called the *dependent* variable, x the *independent* variable.

We will now determine some particular functions of x.

1. The simplest example of $y = f(x)$ is given by

$$y = x.$$

This equation may be plotted by taking a number of values, OA, OB, OC, of x (Fig. 3), and from A, B, C erecting perpendiculars

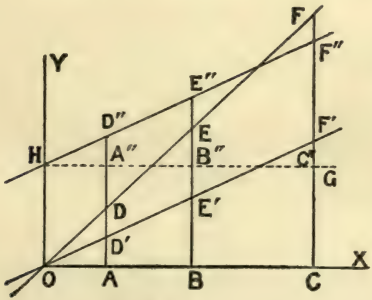

FIG. 3.

$\mathsf{AD} = \mathsf{OA}$, $\mathsf{BE} = \mathsf{OB}$, $\mathsf{CF} = \mathsf{OC}$. Further, when $x = 0$, $y = 0$. Hence, joining ODEF, we get a straight line, inclined to the axis of x at an angle of 45°. (This is obvious, since $\dfrac{y}{x} = \tan \mathsf{FOC} = 1$.)

Hence the equation $y = x$ is graphically represented by the straight line OF, inclined at an angle of 45° to the axis of x.

2. Consider the equation

$$y = ax.$$

By reasoning precisely similar to that used above, this equation is seen to represent a straight line passing through the origin, and inclined to the axis of x at an angle θ, such that $\tan\,\theta = \frac{y}{x} = a$. When a is less than 1, the line is such as OF′.

3. The next degree of complication is given by

$$y = b + ax.$$

This equation may be plotted as follows :—

Let $x = 0$; then $y = b$. Consequently a point on OY, at a distance b from O, will be one point of the curve.*

Now the given equation may also be written

$$y - b = ax.$$

Let OH $= b$, and draw HG parallel to OX. Let $x =$ OA $=$ HA″. Then if A″D″ $= a \times$ HA″ $= ax$, D″ will be a point on the curve. Similarly, E″, F″ will be points on the curve. Also, since

$$\frac{y-b}{x} = a,$$

we see that

$$\frac{D''A''}{HA''} = \frac{E''B''}{HB''} = \frac{F''C''}{HC''} = a = \tan\,\theta, \text{ where } \theta \text{ represents the angle } F''HG.$$

Hence the foregoing equation represents a straight line, passing through the point ($y = b$, $x = 0$), and inclined at an angle θ with OX, where $\tan\,\theta = a$.

4. Any equation involving y and x to the first power only may be put into one of the above forms. Take, for instance, the expression $ax + by + c = 0$.

$$y = -\frac{a}{b}x - \frac{c}{b}.$$

This, therefore, represents a straight line, passing through the point $x = 0$, $y = -\frac{c}{b}$, and inclined at an angle θ to the axis of x. where $\tan\,\theta = -\frac{a}{b}$.

* In the higher mathematics, the term "curve" is applied to all lines, whether curved or straight.

Exercises.—(*a*) Draw the straight line $3x+5y-7=0$, and determine its inclination to the axis of x, and to the straight line $3x-5y+7=0$.

(*b*) A straight line passes through the points $x=3$, $y=5$, and $x=-3$, $y=10$. Determine its equation.

Problems similar to (*b*) are solved as follows:—

We have already found that the equation to any straight line will be of the form

$$ax+by+c=0.$$

To determine the particular straight line under consideration, we must find the value of a, b, and c.

Since the line passes through $x=3$, $y=5$, we have, substituting these values in the above equation,

$$3a+5b+c=0.$$

Similarly, with respect to the second point, we find

$$-3a+10b+c=0.$$

$$\therefore\ 15b+2c=0, \text{ and } b=-\frac{2}{15}c.$$

Also, $$9a+c=0, \text{ and } a=-\frac{1}{9}c.$$

$\therefore$ the required equation may be written

$$-\frac{1}{9}cx-\frac{2}{15}cy+c=0.$$

Dividing through by c, we obtain, after simplifying,

$$5x+6y-45=0.$$

(*c*) A straight line cuts the axis of x at a distance $=+7$ from the origin, and cuts the axis of y at a point distant -4 from the origin. Determine the equation of the line, and find its inclination to the axis of x.

(*d*) A straight line passes through the point $x=3$, $y=7$, and is inclined at an angle of 60° to the axis of x. Find its equation. What is the perpendicular distance of this line from the origin?

We have now examined one of the simplest forms of the equation $y=f(x)$. Just as, in the cases considered, the equations denoted certain straight lines, so, in the general case, $y=f(x)$

denotes some curve. We will now consider some of these curves. Take, for instance,

$$y = \sin x. \quad \text{(See Fig. 4.)}$$

Here x is supposed to be expressed in circular measure. It is well known that the length of the circumference of a circle is proportional to the radius of the circle. Further, the length of an arc of a circle, subtending a given angle at the centre, is propor-

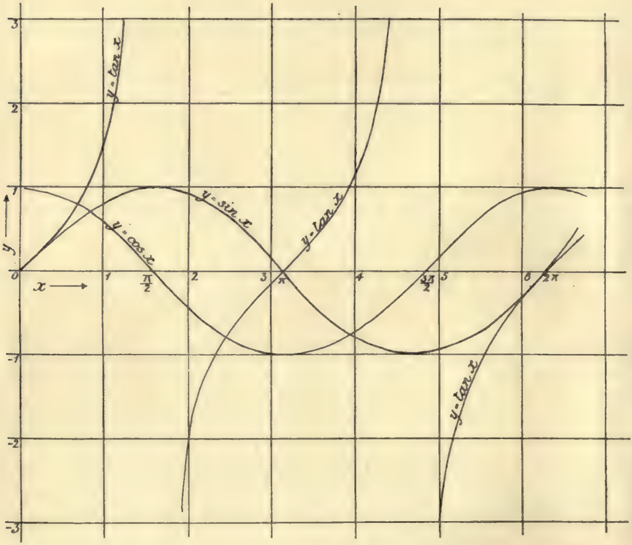

FIG. 4.

tional to the radius of the circle. Hence, for all circular arcs subtending the same angle at the centre of the circle, we have the relation

$$\frac{\text{arc}}{\text{radius}} = \text{constant}.$$

This gives us a method of measuring angles which is universally employed in theoretical investigations. Any two lines meeting at a point being given, with the point of intersection of the lines as centre, draw an arc of a circle cutting the two lines. Then the ratio $\frac{\text{arc}}{\text{radius}}$ is termed the circular measure of the angle.

The unit angle, in this system, is one in which the arc is equal in length to the radius. This unit angle is termed a *radian*. It is equivalent to $57^\circ\ 17'\ 45''$ nearly. The circular measure of $180^\circ = \pi = 3{\cdot}14159$ radians.

To find the length of a given circular arc, multiply the radius of the circle by the circular measure of the angle subtended at the centre.

Let AB, AC (Fig. 5) be two straight lines, meeting each other

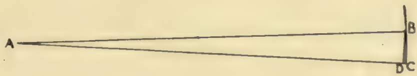

FIG. 5.

at a very acute angle at A. Draw an arc of a circle BC, and from B draw BD perpendicular to AC. Then, if the angle BAC be denoted by x, we have, circular measure of $x = \frac{BC}{AC}$, $\sin x = \frac{BD}{AB} = \frac{BD}{AC}$.

If the angle x is very small, BD will be approximately equal in length to BC. Hence, for small angles, the sine is approximately equal to the circular measure.

It is this property which renders it expedient to express all angles involved in theoretical investigations in circular measure. As an example, the differentiation of trigonometrical functions may be referred to. (See chap. ii., p. 39.)

In Fig. 4, the curves corresponding to the functions

$$y = \sin x,$$
$$y = \cos x, \text{ and}$$
$$y = \tan x,$$

are given. Notice that the last-mentioned curve consists of a number of distinct branches, cutting the axis of x, at angles each equal to 45°, at the points $x = 0$, $x = \pi$, $x = 2\pi$, $x = 3\pi$, etc.

All of the above curves repeat themselves after passing along the axis of x through a distance equal to 2π.

THE CIRCLE.—A circle is defined as a curve such that each point on it is at a certain constant distance from a fixed point, called the centre.

Let us translate this into the language of analysis. Let the centre, C, of the circle be at a point given by $x = a$, $y = b$ (Fig. 6).

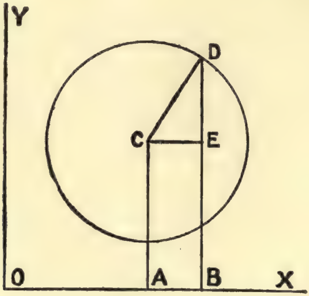

FIG. 6.

Let D be a point on the circle; the radius, CD, being given equal to r. Draw CA, DB perpendicular to OX, and draw CE perpendicular to DB. Then $OA = a$, $AC = b$, $DE = DB - EB = DB - CA = y - b$; and $CE = AB = OB - OA = x - a$. Then since CDE is a right-angled triangle,

$$CE^2 + ED^2 = CD^2. \quad \therefore (x - a)^2 + (y - b)^2 = r^2 \quad . \quad . \quad (1).$$

This is the equation to the circle. It can also be written

$$x^2 + y^2 - 2ax - 2by = r^2 - a^2 - b^2.$$

The following points should be remembered about this equation :—

1. It is an equation involving x and y to the power 2. It is therefore said to be an equation of the *second degree* in x and y.

2. The product xy does not occur.

3. The coefficient of x^2 is equal to the coefficient of y^2, both as to magnitude and sign.

When an equation occurs fulfilling the above three conditions, it will represent a circle.

Exercises.—(*a*) What curve is represented by the equation $3x^2 + 3y^2 + 5x - 9 = 0$? Give full particulars of the curve.

Answer.—This equation may be written

$$x^2+y^2+\frac{5}{3}x-3=0.$$

It therefore represents a circle. It remains to determine the radius, and the position of the centre, when the curve will be completely known.

The above equation may be written

$$x^2+2\frac{5}{6}x+\left(\frac{5}{6}\right)^2+y^2=3+\left(\frac{5}{6}\right)^2, \text{ or}$$
$$\left(x+\frac{5}{6}\right)^2+(y-0)^2=\frac{133}{36}.$$

Comparing this with (1) above, we find that the radius of the circle $=r=\frac{\sqrt{133}}{6}$, and the centre is given by $x=-\frac{5}{6}$, $y=0$.

(*b*) Determine where the above circle cuts the axis of x. (Substitute 0 for y, and solve the resulting quadratic in x.)

(*c*) A circle is drawn through the origin, and the two points $x=0$, $y=5$, and $x=3$, $y=0$. Determine the centre and radius of the circle.

Answer.—Note that three independent constants occur in (1)—namely, a, b, r. Consequently we must have three separate equations to obtain these from—that is, a circle is definitely known when three points through which it passes are known. Here the three points are, the origin ($x=0$, $y=0$), and the points $x=0$, $y=5$, and $x=3$, $y=0$.

The general equation to a circle is of the form

$$x^2+y^2+\mathsf{A}x+\mathsf{B}y+c=0.$$

Hence, substituting in turn the co-ordinates of the three points, we get

1. $(x=0,\ y=0)$, $c=0$.
2. $(x=0,\ y=5)$, $25+5\mathsf{B}+c=0$, or $\mathsf{B}=-5$.
3. $(x=3,\ y=0)$, $9+3\mathsf{A}+c=0$, or $\mathsf{A}=-3$.

$\therefore\ x^2+y^2-3x-5y=0$ is the required equation.

(*d*) Determine the centre and radius of the circle given in (*c*).

(*e*) Write the general equation to a circle in the form of $y=f(x)$.

(From (1) we have $y=b\pm\sqrt{r^2-(x-a)^2}$.)

(f) Write the equation $3x^2+3y^2-12x+6y-60$ in the form $y=f(x)$.

(g) Find the locus of a point such that its distance from the point $(y=0,\ x=3)$ is twice its distance from the point $(y=0,$ $x=-3)$. Ans. $x^2+y^2+10x+9=0$.

(h) A circle, of radius $=5$, is drawn through the two points $(x=1,\ y=0)$ and $x=0,\ y=3$. Determine its equation.

Answer.—Two such circles can be drawn. Their respective equations are

$$(x-5)^2+(y-3)^2=25$$

and

$$(x+4)+y^2=25.$$

(i) Find the co-ordinates of the points of intersection of the circle $(x-3)^2+(y-4)^2=9$ and the straight line

$$y=1+\frac{4}{3}x.$$

(Substitute the value of y in terms of x (given by equation of straight line), in the equation to the circle. A quadratic in x is thence obtained. Substitute the two values of x, obtained by solving the quadratic, in the equation of the straight line, and the corresponding values of y will be obtained.)

The Ellipse.

Problem.—A point moves so that the sum of its distances from two fixed points remains constant. Determine the locus of the movable point.

Take the line through the two fixed points as the x axis, the y axis being drawn perpendicular to this line, midway between the two points. Let the distance from the origin to either of the fixed points $=\delta$.

Let P (Fig. 7) be a point on the curve, A and B being the two fixed points, and draw PD perpendicular to the x axis. Then

$$\mathrm{PB}^2=\mathrm{PD}^2+\mathrm{DB}^2,$$
$$\mathrm{PA}^2=\mathrm{PD}^2+\mathrm{AD}^2.$$

Also, since $\mathrm{PD}=y$, $\mathrm{OD}=x$, $\mathrm{DB}=\delta-x$, $\mathrm{AD}=\delta+x$, we shall have, if the sum of the distances of P from A and B is equal to the constant value c,

$$\{(\delta+x)^2+y^2\}^{\frac{1}{2}}+\{(\delta-x)^2+y^2\}^{\frac{1}{2}}=c.$$

$$\therefore\ (\delta+x)^2+y^2=\left[c-\{(\delta-x)^2+y^2\}^{\frac{1}{2}}\right]^2.$$

$$\delta^2+2\delta x+x^2+y^2=c^2-2c\{(\delta-x)^2+y^2\}^{\frac{1}{2}}+\delta^2-2\delta x+x^2+y^2.$$

Hence $c^2-4\delta x=2c\sqrt{\{(\delta-x)^2+y^2\}}$.

Squaring both sides, we get

$$\begin{aligned}c^4-8c^2\delta x+16\delta^2x^2&=4c^2\{(\delta-x)^2+y^2\}\\&=4c^2(\delta^2-2\delta x+x^2+y^2).\end{aligned}$$

Collecting terms in x^2 and y^2,

$$4x^2(c^2-4\delta^2)+4c^2y^2=c^2(c^2-4\delta^2).$$

$$\therefore\ \frac{x^2}{\frac{c^2}{4}}+\frac{y^2}{\frac{c^2-4\delta^2}{4}}=1 \quad . \quad . \quad . \quad . \quad . \qquad (1).$$

This is the equation to the curve in terms of the constants given in the problem.

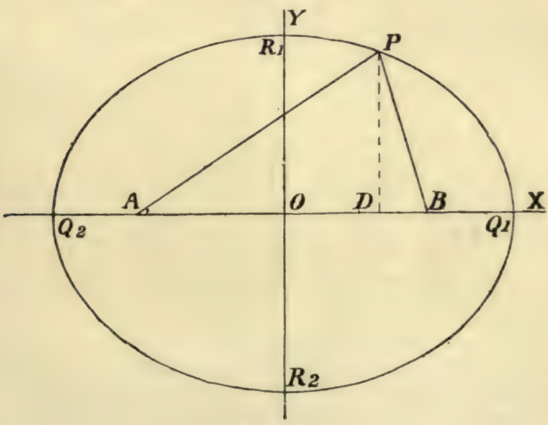

FIG. 7.

Let $y=0$. Then the two values obtained for x will give the points where the curve cuts the axis of x.

$$x=\pm\frac{c}{2}=OQ_1 \text{ and } OQ_2. \quad \text{(Fig. 7.)}$$

Similarly, let $x=0$. Then

$$y=\pm\frac{\sqrt{c^2-4\delta^2}}{2}=\mathsf{OR}_1 \text{ and } \mathsf{OR}_2.$$

The student should verify these values of OQ and OR from first principles.

The curve obtained is called an *ellipse*, of which A and B are the two *foci*. OQ_1 is called the *semi-axis major*, and OR_1 is called the *semi-axis minor.*

Let $\mathsf{OQ}_1=a$, $\mathsf{OR}_1=b$. Then the equation to the ellipse may be written

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad . \quad . \quad . \quad . \quad . \quad . \qquad (2).$$

Example 1.—The major and minor axes of an ellipse are respectively 20 centimetres and 10 centimetres in length: find the equation of the ellipse, also the distance from the origin to a focus.

Ans. $\frac{x^2}{100}+\frac{y^2}{25}=1.$

Distance of focus from origin $=\sqrt{75}$ centimetres.

(The latter follows from the equations

$$\frac{c}{2}=10, \text{ and } \frac{\sqrt{c^2-4\delta^2}}{2}=5.)$$

Example 2.—Write the equation of the ellipse in the form $y=f(x)$.

Ans. $y=\pm b\sqrt{1-\frac{x^2}{a^2}}$, or $\pm\sqrt{b^2-\frac{b^2}{a^2}x^2}$, or $\pm\frac{b}{a}\sqrt{a^2-x^2}$.

The Hyperbola.

Problem.—A point moves so that the *difference* between its distances from two fixed points remains constant. Find the locus of the movable point.

The equation may be found, in a manner similar to that used above, to be

$$\frac{x^2}{\frac{c^2}{4}}-\frac{y^2}{\frac{4\delta^2-c^2}{4}}=1.$$

Transform as above to the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$

This curve is called a *hyperbola*. It consists of two branches, on opposite sides of the axis of y. In Fig. 8, the axis OX is drawn

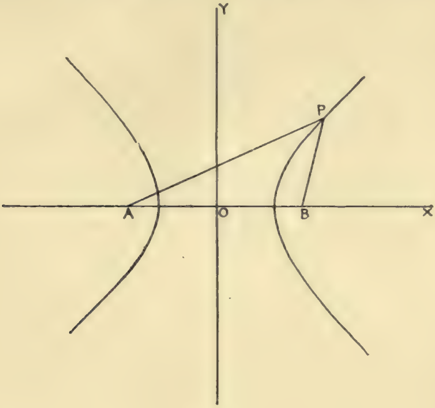

Fig. 8.

through the two fixed points A, B, the axis OY being drawn perpendicular to OX, through the point O, midway between A and B. The difference between the distances PA, PB, of any point P, from the foci A and B, is the same wherever the point P may be situated on the curves. On the right-hand curve, $PA > PB$; whilst on the left-hand curve, $PB > PA$.

The Parabola.

Problem.—A point moves so that its distance from a fixed point is equal to its perpendicular distance from a given straight line. Find the locus of the movable point.

Take, as the axis of x, the line AB (Fig. 9), through the fixed point A, and perpendicular to the given straight line BE. Bisect AB in C. Then it is easily seen that C will be a point on the curve. Draw the straight line CY perpendicular to the line BD, and passing through to point C, and let CY be taken as the axis of y, and CD as the axis of x. Also let $CA = CB = a$.

Take any point, P, on the curve, and draw PD perpendicular to the axis of x. Then

$$AP^2 = AD^2 + DP^2.$$

Draw PQ perpendicular to BE. Then, by the condition of the problem,

$$PA = PQ = BD = BC + CD = a + x.$$
$$AD = CD - CA = x - a, \text{ and } DP = y.$$

Then $$\{(x-a)^2 + y^2\}^{\frac{1}{2}} = a + x.$$
Squaring, $$x^2 - 2ax + a^2 + y^2 = a^2 + 2ax + x^2.$$
$$\therefore\ y^2 = 4ax.$$

This is the equation of the required locus.

The above curve is termed a *parabola*, of which A is the *focus*, and BE is the *directrix*. The point C is termed the *vertex*.

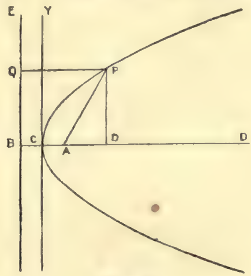

Fig. 9.

The equation $x^2 = 4ay$ will, similarly, represent a parabola, passing through the origin, and having a focus at the point $y = a$, $x = 0$.

Exponential Functions.—It remains to consider a class of functions of the greatest importance, termed *exponential functions.* These functions are commonly treated of in books on algebra and trigonometry. A brief consideration of them will be given here, as the student's attention must be directed particularly to one or two points.

Let it be required to expand a^x in a series of ascending powers of x.

(It is assumed that the student is acquainted with the binomial theorem that

$$(a+x)^n = a^n + na^{n-1}x + \frac{n(n-1)}{1 \cdot 2} a^{n-2}x^2$$
$$+ \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}x^3 + \ldots.$$

This theorem will be frequently used during the development of the subject.)

We may write

$$a^x = \left(1 + \overline{a-1}\right)^x$$

$$= 1 + x(a-1) + \frac{x(x-1)}{1\cdot 2}(a-1)^2$$

$$+ \frac{x(x-1)(x-2)}{1\cdot 2\cdot 3}(a-1)^3 + \frac{x(x-1)(x-2)(x-3)}{1\cdot 2\cdot 3\cdot 4}(a-1)^4 + \ldots, \text{ etc.}$$

Rearrange this expression in a series of ascending powers of x.

$$a^x = 1 + x\left\{(a-1) - \frac{1}{1\cdot 2}(a-1)^2 + \frac{1\cdot 2}{1\cdot 2\cdot 3}(a-1)^3 - \frac{1\cdot 2\cdot 3}{1\cdot 2\cdot 3\cdot 4}(a-1)^4 + \ldots\right\} + \text{ terms in } x^2,\ x^3, \text{ etc.}$$

The required series may therefore be written

$$a^x = 1 + \mathsf{A}x + \mathsf{B}x^2 + \mathsf{C}x^3 + \mathsf{D}x^4 + \ldots \qquad (3),$$

where

$$\mathsf{A} = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 + \ldots \qquad (4).$$

We shall now proceed to determine B, C, D, etc.

Since a^y is obtained from a^x by simply changing x into y, we see that

$$a^y = 1 + \mathsf{A}y + \mathsf{B}y^2 + \mathsf{C}y^3 + \mathsf{D}y^4 + \ldots \qquad (5).$$

According to the theory of indices,

$$a^{(x+y)} = a^x \times a^y.$$

Hence,

$$a^{(x+y)} = 1 + \mathsf{A}(x+y) + \mathsf{B}(x+y)^2 + \mathsf{C}(x+y)^3 + \mathsf{D}(x+y)^4$$
$$= a^x \times a^y = (1 + \mathsf{A}x + \mathsf{B}x^2 + \mathsf{C}x^3 + \mathsf{D}x^3 + \ldots)$$
$$\times (1 + \mathsf{A}y + \mathsf{B}y^2 + \mathsf{C}y^3 + \mathsf{D}y^4 + \ldots).$$

Multiplying together the two series whose product $= a^x \cdot a^y \cdot$, we get

$$a^x \cdot a^y = 1 + \mathsf{A}(x+y) + \{\mathsf{B}(x^2+y^2) + \mathsf{A}^2xy\} + \{\mathsf{C}(x^3+y^3) + \mathsf{AB}(x^2y+xy^2)\} + \ldots$$

Since this series must be, term for term, identical with the series for $a^{(x+y)}$, and since the first two terms of both series are identical as they stand, we get

$$B(x+y)^2 = B(x^2+y^2+2xy) = B(x^2+y^2) + A^2xy.$$

Hence, $$2B = A^2\,;\ \therefore\ B = \frac{A^2}{2}\cdot$$

Similarly,

$$C(x^3+y^3+3x^2y+3xy^2) = C(x^3+y^3) + AB(x^2y+xy^2).$$

$$\therefore\ 3C = AB\,;\ \therefore\ C = \frac{AB}{3} = \frac{A^3}{2\cdot 3}\cdot$$

Similarly, it may be shown that

$$D = \frac{A^4}{1\cdot 2\cdot 3\cdot 4},\ E = \frac{A^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}\cdot$$

Finally,

$$a^x = 1 + Ax + \frac{A^2}{1\cdot 2}x^2 + \frac{A^3}{1\cdot 2\cdot 3}x^3 + \frac{A^4}{1\cdot 2\cdot 3\cdot 4}x^4 + \ldots \qquad (6),$$

where A has the value given in (4).

Let us examine the case when $A = 1$; that is, when the series $(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 - \ldots = 1$. Let the value of a, which will make this series $= 1$, be denoted by ϵ. Then

$$\epsilon^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \ldots \qquad (7).$$

To find ϵ, put $x = 1$. Then

$$\epsilon(=\epsilon^1) = 1 + 1 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \ldots$$

Exercise.—Calculate the value of ϵ correctly to five places of decimals.

Answer.—$\epsilon = 2{\cdot}71828$.

Now, whatever the value of a may be, it may obviously be written ϵ^z, if a suitable value be found for z; and since z here indicates the power to which ϵ must be raised in order to equal a, z may be termed the logarithm* of a to the base ϵ, or the Napierian† logarithm of a, and written as $\log_\epsilon a$.

Then $$a^x = (\epsilon^z)^x = (\epsilon^{\log_\epsilon a})^x = \epsilon^{(\log_\epsilon a\,.\,x)} = 1 + \log_\epsilon a\cdot x + \frac{(\log_\epsilon a)^2}{\lfloor 2}x^2 + \frac{(\log_\epsilon a)^3}{\lfloor 3}x^3 + \ldots \qquad \text{from (7).}$$

---

* Logarithm, literally the *number of the ratio*, from Gr. *logos*, ratio, and *arithmos*, number.

† Napier was the discoverer of the properties of logarithms to the base ϵ.

Comparing this result with (6), we see that

$$\log_\epsilon(a) = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \tfrac{1}{4}(a-1)^4 \; . \; . \; . \; . \quad (8).$$

As we proceed with the study of the calculus, it will become clear that the theoretical importance attaching to logarithmic and exponential functions is due to certain properties of the series given in (7) and (8).

Let $$y = a^x = \epsilon^{\log_\epsilon a \cdot x}.$$

Then $$\log_a y = x = x \cdot \log_\epsilon a \cdot \log_a \epsilon.$$

$$\therefore \; \log_\epsilon a = \frac{1}{\log_a \epsilon},$$

$$\log_\epsilon y = \log_\epsilon a \cdot x = \log_\epsilon a \cdot \log_a y.$$

Exercise.—Find the value of $\log_\epsilon 5$.

Answer.—$\mathrm{Log}_\epsilon 5 = \log_\epsilon 10 \cdot \log_{10} 5 = \dfrac{\log_{10} 5}{\log_{10} \epsilon} = 1{\cdot}609 \; . \; . \; . \; .$

Hence, to convert common logarithms (base 10) to Napierian logarithms (base ϵ), we multiply by $2{\cdot}3026$, or divide by $\cdot 43429$.

Exercise.—Plot the curve $y = \epsilon^x$ for values of x between 0 and 2.

(Substitute 0, ·25, ·5, ·75, 1, 1·5, and 2 for x in (7), calculating your results to two places of decimals.)

With the aid of a table of common logarithms, the values of ϵ^x may also be calculated as follows :—

$$\epsilon^\circ = 1.$$

To find the value of $\epsilon^{\cdot 25}$.

$$\epsilon = 2{\cdot}718 \; . \; . \; . \; .$$

$$\log_{10} 2{\cdot}718 = \cdot 4343$$

$$\log_{10} \epsilon^{\cdot 25} = \cdot 25 \; \log_{10} \epsilon = \tfrac{1}{4} \; \log_{10} \epsilon.$$

$$\therefore \; \log_{10} \epsilon^{\cdot 25} = \cdot 1086.$$

$$\therefore \; \epsilon^{\cdot 25} = 1{\cdot}284.$$

Exercise.—Plot the curve $y = \epsilon^{-x}$ for values of x between 0 and 2.

Exercise.—Plot the curves $y = \dfrac{\epsilon^x + \epsilon^{-x}}{2}$ and $y = \dfrac{\epsilon^x - \epsilon^{-x}}{2}$.

(Note that

$$\frac{\epsilon^x + \epsilon^{-x}}{2} = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \ldots,$$

a series involving only *even* powers of x.

On the other hand,

$$\frac{\epsilon^x - \epsilon^{-x}}{2} = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \ldots,$$

a series involving only *odd* powers of x.)

We have given a brief consideration to those functions which are most likely to occur in physical investigations. We are now in a position to commence to develop the principles of the calculus as applied to these functions.

CHAPTER II.

DIFFERENTIALS, AND DIFFERENTIAL COEFFICIENTS.

TANGENTS.—Let ABC (Fig. 10) represent a part of the curve corresponding to the analytical expression $y = f(x)$. Let any point B be taken on this curve, and let a straight line BD_1 be

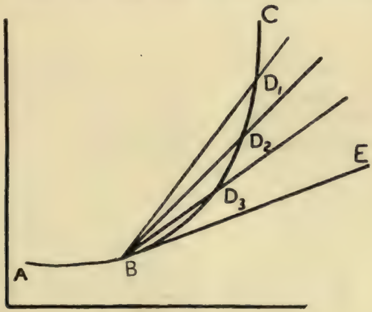

FIG. 10.

drawn through it and any other point D_1 on the curve. If now successive points, D_2, D_3, D_n, are taken between D_1 and B, it will be seen that the straight line drawn through B and any one of these points, D_n, will approach the limiting position BE as D_n is taken nearer and nearer to B. BE is termed the tangent to the curve ABC at the point B. It may be defined as *a straight line passing through* B *and another point on the curve infinitely close to* B.

The student must carefully distinguish between the terms *tangent to a curve* (which has the meaning above assigned) and the *tangent of an angle*, which represents the trigonometrical ratio $\frac{\text{perpendicular}}{\text{base}}$ for a right-angled triangle.

Let us now see what procedure may be employed in order to determine the inclination of the geometrical tangent at any point of a curve with regard to the axis of x.

Let A (Fig. 11) be the given point on the curve whose equation

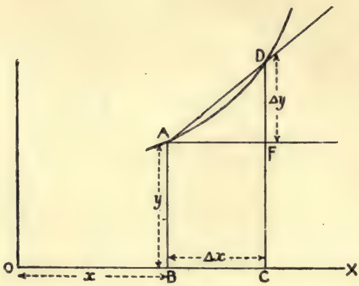

FIG. 11.

is $y = f(x)$. Then if we substitute the length OB for x in this equation, we shall obtain the value of y corresponding to AB.

Now take another point, C, on the x axis, and let BC be denoted by Δx.* Draw CD through C perpendicular to OX, cutting the given curve in D. Then the length DC will be obtained by substituting the length OC—that is, $x + \Delta x$—in the equation of the curve. Draw AF parallel to OX. Then, since FC = AB = y, we may represent FD by Δy. Thus Δy represents the increment of y corresponding to the increment Δx of x; or, to put it in different words, when the value of x is increased by Δx, the value of y is increased by Δy.

The trigonometrical tangent to the angle DAF $= \frac{\text{DF}}{\text{AF}} = \frac{\Delta y}{\Delta x}$, as above defined. Hence $\frac{\Delta y}{\Delta x}$ represents the trigonometrical tangent of the angle of inclination of AD to the axis of x. If D is supposed to approach indefinitely near to A, the line AD will approximate to the tangent at A to the given curve. But as the length AD becomes indefinitely small, both DF and AF—that is, Δy and Δx—will become indefinitely small. What we are concerned with, however, is the *ratio* of $\frac{\Delta y}{\Delta x}$; and though both numerator and denominator diminish in value, the above ratio will generally tend toward a finite limiting value. This limiting value will give *the*

* The student should particularly notice that the symbol Δx, *as a whole*, is here used to represent a small increase in length. Δx is *not* equivalent to $\Delta \times x$.

tangent of the angle that the geometrical tangent to the curve, at the point A, makes with the axis of x.

Now we have

$$y+\Delta y=f(x+\Delta x),$$
$$y=f(x).$$
$$\therefore\ \Delta y=f(x+\Delta x)-f(x),$$
$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

If we now let Δx decrease indefinitely—that is, approach a value of zero—we may rewrite the above equation

$$\frac{dy}{dx}=\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}\right)\text{ when }\Delta x=0.$$

$\frac{dy}{dx}$ is thus equal in value to the tangent of the angle that the geometrical tangent at the point (x, y) of the curve makes with the axis of x. $\frac{dy}{dx}$ is generally called *the differential coefficient of y with respect to x.*

A much clearer notion of the exact purport of the above reasoning will be gained by taking a few simple examples. Take the equation $y=a+bx$, which was shown on page 12 to represent a straight line passing through the point $(y=a, x=0)$, and inclined at an angle, whose tangent is b, to the axis of x.

$$y=a+bx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$y+\Delta y=a+b(x+\Delta x).$$
$$\therefore\ \Delta y=a+b(x+\Delta x)-y=a+b(x+\Delta x)-(a+bx)=b\Delta x,$$
$$\frac{\Delta y}{\Delta x}=\frac{b\Delta x}{\Delta x}=b.$$
$$\therefore\ \frac{dy}{dx}=b.$$

Since the particular curve represented by (1) is a straight line, the tangent to this curve—that is, a straight line passing through two very nearly coincident points on (1)—is a straight line coinciding with (1). The interpretation of the equation $\frac{dy}{dx}=b$ may be left to the student, who should carefully re-read the foregoing part of the present chapter.

Take the equation $x^2=4ay$. This represents, as explained in the preceding chapter, a parabola passing through $(x=0)$, $(y=0)$, and having a focus at a distance from the origin $=a$, along the axis of y. The general form of this curve is given in Fig. 12.

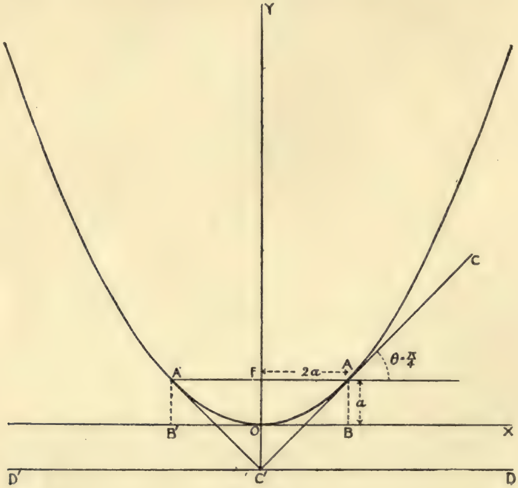

FIG. 12.

$$y=\frac{1}{4a}x^2.$$

$$y+\Delta y=\frac{1}{4a}(x+\Delta x)^2$$

$$=\frac{1}{4a}(x^2+2x\cdot\Delta x+(\Delta x)^2).$$

$$\Delta y=\frac{1}{4a}(x^2+2x\Delta x+(\Delta x)^2)-y=\frac{1}{4a}\{x^2+2x\Delta x+(\Delta x)^2-x^2\}$$

$$=\frac{1}{4a}\cdot(2x\Delta x+(\Delta x)^2).$$

$$\therefore\ \frac{\Delta y}{\Delta x}=\frac{1}{4a}\cdot\{2x+\Delta x\} \quad . \quad . \quad . \quad (2).$$

The latter expression is obtained by dividing through by Δx. Now let us suppose that we decrease Δx indefinitely. The first term on the right-hand side of (2) is independent of Δx; but the second term—namely, $\frac{1}{4a}\Delta x$—can be made as small as we please by decreasing the value of Δx. Hence when Δx is made infinitely small, and, as a consequence, we write $\frac{dy}{dx}$ instead of $\frac{\Delta y}{\Delta x}$, (2) becomes

$$\frac{dy}{dx}=\frac{2x}{4a}+0=\frac{1}{2a}x.$$

This means that if we take any point, A, on the parabola, Fig. 12, so that $OB=x$, the inclination θ of the tangent AC to the line OX (or FA) is determined from the equation

$$\frac{dy}{dx}=\tan\theta=\frac{1}{2a}x.$$

As the whole of the reasoning connected with both the differential and integral calculus depends on the method just employed, it is well here to carefully note the exact procedure followed.

We are given the equation $y=f(x)$. From this, by the above train of reasoning, we deduce

$$\frac{\Delta y}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

when Δx and Δy denote the corresponding finite increments of x and y. We now have to expand the expression $f(x+\Delta x)$ in a series of ascending powers of Δx, and subtract $f(x)$ from this series. We then divide through by Δx, and finally put $\Delta x=0$, when we obtain the value of $\frac{dy}{dx}$.

It should be noticed that in (2) we neglect the second term on the right-hand side, since it is infinitely small *in comparison with the first term*, which we retain. Further, the student will have to slightly modify his ideas of the meaning of 0, or zero, in order to follow the developments of the higher mathematics. In ordinary arithmetic and elementary algebra, 0 denotes *no-thing*, and as long as we are dealing with mere numbers that definition is sufficiently accurate. Thus, $3-3=0$, if the 3 in each case denotes (for example) apples or pennies. But if we speak of subtracting one length from another (that is, 3 yards – 3 yards), another meaning may be given to the result. For instance, if both magnitudes are measured with scales correct, say, to one-tenth of an inch, but each involving a possible error of measurement which may amount to several hundredths of an inch, we might write the above equation 3 measured yards − 3 measured yards = a quantity not greater than $\frac{1}{10}$ inch, which may be neglected as being beyond our power to measure with the given graduated scales.

A simple example of the meaning of neglecting a certain value in comparison with another value may be given. A man whose income amounted to some thousands of pounds would not count any odd halfpence in his possession at the end of the year. He would neglect the halfpence in comparison with the pounds. On the other hand, to a beggar, whose capital amounted to $2\frac{3}{4}$d., a halfpenny might have a value far from negligible.

To return now to the parabola. Consider the following problems:—

(1.) At what points on the parabola given by the equation $x^2=4ay$ will the tangent make angles with the axis of x equal to $0°$, $+45°$, and $-45°$ respectively.

The trigonometrical tangent of the angle θ which the geometrical tangent to the parabola at a point (x, y) makes with the axis of x is given by

$$\tan\theta=\frac{dy}{dx}=\frac{1}{2a}x,$$

$$\text{when } \theta=0, \tan\theta=0, \text{ and therefore } \frac{dy}{dx}=0.$$

$$\therefore \frac{1}{2a}x=0.$$

The only value of x which will satisfy the above expression is $x=0$. Further, if 0 be substituted for x in the equation to the parabola, it will be seen that $y=0$. Therefore the tangent at the origin to the above parabola will coincide with the axis of x.

$$\text{When } \theta=45°, \tan\theta=1.$$

$$\therefore \frac{1}{2a}x=1,$$

$$x=2a.$$

Substituting this value of x in the equation

$$x^2=4ay,$$

we get

$$(2a)^2=4a^2=4ay.$$

$$\therefore y=a.$$

Therefore the tangent to the above curve at the point $(y=a)$, $(x=2a)$ will make an angle of 45° with the axis of x.

When $\theta=-45^\circ$, $\tan\theta=-1$.

$$\frac{1}{2a}x=-1,$$

$$x=-2a.$$

Also, $$(-2a)^2=4a^2=4ay.$$

$$\therefore\ y=a.$$

Hence the tangent to the parabola at the point $(y=a)$, $(x=-2a)$ will make an angle of -45° with the axis of x.

These two tangents are shown as AC, A′C′ in Fig. 12. Notice that as we pass from A′ round to A along the curve, the negative angle (which the tangent at all points where x is negative makes with the x axis) diminishes and becomes zero when $x=0$; whilst the angle which the tangent makes with the x axis subsequently becomes positive, and increases continuously. It should be remembered that a numerical decrease in the value of a *negative* angle is really an algebraical increase in the value of the angle.

The foregoing results exhibit some well-known properties of a parabola. The co-ordinates of A are

$$x=\mathrm{OB}=2a,$$
$$y=\mathrm{BA}=a.$$

Similarly, the co-ordinates of A′ are

$$x=\mathrm{OB'}=-2a,$$
$$y=\mathrm{B'A'}=a.$$

Join AA′. Then this straight line passes through the focus F of the parabola, since the latter is situated at a distance equal to a, measured in the positive direction along the axis of y. The line AA′ is termed the *latus rectum* of the parabola.

Produce the tangent AC backwards, till it meets the tangent A′C′ in C′. Then it is obvious, from the symmetry of the curve, that C′ is on the axis of y, produced in the negative direction.

Also, since the angle $\mathrm{FAC'}=45^\circ$, and the angle $\mathrm{AFC'}=90^\circ$, the angle $\mathrm{FC'A}=45^\circ$, the triangle FAC′ is isosceles, and $\mathrm{FC'}=\mathrm{FA}=2a$. But

$$\mathrm{OF}=a,$$
$$\therefore\ \mathrm{OC'}=a.$$

Hence the straight line D′C′D, drawn through C′, parallel to the axis of x, is the *directrix* of the parabola. (See page 22.) We thus obtain the result that *the tangents at the extremities of the latus rectum of a parabola intersect at right angles on the directrix.*

(2.) Find the equation of the tangent to the parabola $x^2=4ay$, at a point whose abscissa is x_1.

(*a*) The tangent will be a straight line; hence, as shown previously, its equation will have the form $y=\mathsf{A}+\mathsf{B}x$.

(*b*) Since B denotes the tangent of the angle made by this straight line with the x axis, and since the geometrical tangent to the curve, at a point whose abscissa is equal to x_1, is found by substituting this value of x in $\frac{dy}{dx}$, where

$$\frac{dy}{dx}=\frac{1}{2a}\cdot x,$$

we have
$$\mathsf{B}=\frac{1}{2a}x_1.$$

$$\therefore\ y=\mathsf{A}+\left(\frac{1}{2a}x_1\right)x \quad . \quad . \quad . \quad (3)$$

will be the equation to the tangent when the proper value of A is substituted.

(*c*) The tangent passes through the point $\left(y=\frac{1}{4a}x_1^2\right)$, $(x=x_1)$. Substituting these values in (3), we get

$$\frac{1}{4a}x_1^2=\mathsf{A}+\frac{1}{2a}x_1^2.$$

$$\therefore\ \mathsf{A}=-\frac{1}{4a}x_1^2.$$

Finally, the required equation of the tangent is given by substituting this value of A in (3). Hence we get

$$y=-\frac{1}{4a}x_1^2+\frac{1}{2a}x_1x.$$

(3.) At what point will the above tangent cut the axes of x and y respectively?

Put $x=0$, then $y=-\frac{1}{4a}x_1^2$.

Put $y=0$, then $x=\frac{1}{2}x_1$.

(4.) Show that tangents to the above parabola, at points whose abscissæ are $+x_1$ and $-x_1$ respectively, will intersect in the axis of y.

(5.) At what point will the above tangent to the parabola intersect the directrix?

The equation to the directrix is easily seen to be $y=-a$. Substitute this value in equation to tangent, and we get

$$x=\frac{x_1}{2}-\frac{2a^2}{x_1}.$$

Differentiation.

We can now proceed systematically to differentiate the most important simple functions met with in physical investigations.

I. $y=ax^n$ . . . . (1).

Then $y+\Delta y=$

$$a(x+\Delta x)^n=ax^n+nax^{n-1}\Delta x+\frac{n(n-1)}{1\cdot 2}ax^{n-2}(\Delta x)^2+\ \ldots,$$

expanding by the binomial theorem.

$$\therefore\ \Delta y=nax^{n-1}\Delta x+\frac{n(n-1)}{1\cdot 2}ax^{n-2}(\Delta x)^2+\ \ldots \quad (2).$$

When Δx is taken infinitely small, and is then written dx, we must alter Δy into dy. Notice that the second term on the right-hand side of (2) is multiplied by the square of Δx, whilst the first term is only multiplied by the first power of Δx. Consequently, since dx is indefinitely small, the series will be convergent (that is, the value of any term is much less than that of the preceding term), and the term in $(dx)^2$ may be neglected in comparison with the term in dx, and so on. Hence

$$dy=nax^{n-1}dx \quad . \quad . \quad . \quad . \quad (3).$$

$$\frac{dy}{dx}=nax^{n-1} \quad . \quad . \quad . \quad . \quad (4).$$

We have already seen how a definite meaning may be attached to the value of $\frac{dy}{dx}$ in (4). Thus, if θ be the angle that the tangent to the curve $y=ax^n$, at the point (x, y), makes with the axis of x, then

$$\frac{dy}{dx}=\tan\theta=nax^{n-1} \quad . \quad . \quad . \text{ by (4).}$$

Hence, from (3), $\quad dy=dx\tan\theta.$

The meaning of this may be seen on referring to Fig. 13. AB shows the general form of that part of the curve, represented by (1) (n being greater than unity), which corresponds to the positive value of x. Let $AC=x$, $CD=dx$, $CE=y$. If EG is drawn parallel to the axis of x, then $EG=dx$, $FG=dy$. Then, as EG is diminished indefinitely, the curved line EF will approximate more and more

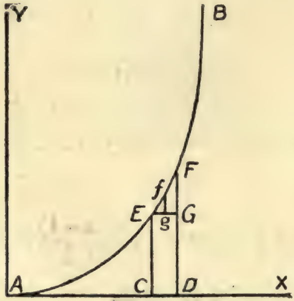

FIG. 13.

closely to a straight line, the hypotenuse of the right-angled triangle EFG. In that case, $FG=dy=EG \tan FEG=dx \tan \theta$ $=\frac{dy}{dx}\cdot dx$, . . . Equation (3) expresses the value of the *differential* of y, whilst (4) gives the value of the *differential coefficient* of y with respect to x.

It may be noticed that it is not permissible, without further consideration, to cancel the dx's on the right-hand side of the equation

$$dy=\frac{dy}{dx}\cdot dx \quad . \quad . \quad . \quad (5).$$

For $\frac{dy}{dx}$ is the ultimate value of the ratio $\frac{\Delta y}{\Delta x}$, when Δy and Δx are diminished indefinitely. Now dy on the left of (5), and dx on the extreme right of the same equation, are of course supposed to be indefinitely small; but neither need have the same values as the corresponding quantities in $\frac{dy}{dx}$. Thus, in Fig. 1, $\frac{dy}{dx}$ may be represented by the ratio $\frac{fg}{Eg}$, whence we shall have the relation

$$dy = \mathsf{FG} = \mathsf{EG} \tan \theta = \mathsf{EG} \frac{fg}{\mathsf{E}g} = \frac{dy}{dx} \cdot dx,$$

the only condition being that in all cases dy and dx (and therefore FG and EG, fg and $\mathsf{E}g$) are indefinitely small.

Equation (4) will be true whatever value may be given to n, whether positive or negative, integral or fractional. The rule for finding the *differential coefficient* of an expression consisting of a constant multiplied by some power of x may therefore be formulated as follows: *Write down the given function with the index of x diminished by unity, and multiply the result by the original index of x.*

Some examples will make this clearer :—

(1.) $y = 3x^4$, then $\dfrac{dy}{dx} = 4 \times 3x^{(4-1)} = 12x^3.$

(2.) $y = 5x^{-4}$, then $\dfrac{dy}{dx} = (-4) \times 5x^{(-4-1)} = -20x^{-5}.$

(3.) $y = 7x^{\frac{3}{2}}$, then $\dfrac{dy}{dx} = \frac{3}{2} \times 7x^{\left(\frac{3}{2}-1\right)} = \frac{21}{2}x^{\frac{1}{2}}.$

(4.) $y = x^{\frac{1}{3}}$, then $\dfrac{dy}{dx} = \frac{1}{3}x^{\left(\frac{1}{3}-1\right)} = \frac{1}{3}x^{-\frac{2}{3}}.$

(5.) $y = 3x^{-\frac{2}{5}}$, then $\dfrac{dy}{dx} = (-\frac{2}{5}) \times 3x^{\left(-\frac{2}{5}-1\right)} = -\frac{6}{5}x^{-\frac{7}{5}}.$

It will form an instructive exercise for the student to work out each of the above examples, first using the rule given above—which expresses in words the result arrived at in equation (4)—and then obtaining the differential coefficient from first principles.

It will be remarked that a special example of the equation

$$y = ax^n,$$

where n had an integral value equal to 2, has already been considered. Only one other example will here be worked out from first principles—namely,

$$y = 7x^{-\frac{1}{5}}.$$

$$y + \Delta y = 7(x + \Delta x)^{-\frac{1}{5}} = 7x^{-\frac{1}{5}} + (-\tfrac{1}{5}) \times 7x^{\left(-\frac{1}{5}-1\right)}\Delta x$$
$$+ \frac{(-\frac{1}{5})(-\frac{1}{5}-1)}{1 \cdot 2} 7x^{\left(-\frac{1}{5}-2\right)}(\Delta x)^2 + \ldots, \text{etc.}$$

$$\therefore \Delta y = -\tfrac{7}{5}x^{-\frac{6}{5}}(\Delta x) + \tfrac{42}{50}x^{-\frac{11}{5}}(\Delta x)^2 + \ldots$$

When Δx is made infinitely small, this may be written—

$$dy = -\tfrac{7}{5}x^{-\frac{6}{5}}dx,$$
$$\frac{dy}{dx} = -\tfrac{7}{5}x^{-\frac{6}{5}}.$$

(Before working the examples to follow, the student is advised to carefully re-read the section in the preceding chapter devoted to logarithmic and exponential functions.)

II. $$y = a^{bx}.$$

This may be written $y = \epsilon^{b \log_\epsilon a . x}$, so as to reduce to the general form

$$y = \epsilon^{cx}.$$
$$y + \Delta y = \epsilon^{c(x + \Delta x)} = \epsilon^{(cx + c\Delta x)} = \epsilon^{cx} . \epsilon^{c\Delta x},$$

according to the convention respecting exponentials.

$$\therefore \Delta y = \epsilon^{cx} \cdot \epsilon^{c\Delta x} - \epsilon^{cx} = \epsilon^{cx}(\epsilon^{c\Delta x} - 1).$$

But $$\epsilon^{c\Delta x} = 1 + c\Delta x + \frac{(c\Delta x)^2}{1 \cdot 2} + \frac{(c\Delta x)^3}{1 \cdot 2 \cdot 3} + \ldots .$$

(See page 24.)

Hence $$\Delta y = \epsilon^{cx}\left(c\Delta x + \frac{(c\Delta x)^2}{1 \cdot 2} + \frac{(c\Delta x)^3}{1 \cdot 2 \cdot 3} + \ldots \right) \quad (6).$$

When Δx is made infinitely small, the series within the brackets will be convergent, and the terms involving the square or higher powers of Δx may be neglected in comparison with the term in Δx. Hence, altering Δy and Δx into dy and dx,

$$dy = c\epsilon^{cx}dx,$$
$$\frac{dy}{dx} = c\epsilon^{cx}.$$

Hence if $$y = \epsilon^x,$$
$$\frac{dy}{dx} = \epsilon^x.$$

If $$y = a^{bx} = \epsilon^{b \log_\epsilon a . x},$$
$$dy = b \log_\epsilon a \cdot \epsilon^{b \log_\epsilon a . x}\, dx = b \log_\epsilon a \cdot a^{bx}dx,$$
$$\frac{dy}{dx} = b \log_\epsilon a \cdot a^{bx}.$$

III. $$y = \log_\epsilon x.$$

$$y + \Delta y = \log_\epsilon(x + \Delta x) = \log_\epsilon x\left(1 + \frac{\Delta x}{x}\right) = \log_\epsilon x + \log_\epsilon\left(1 + \frac{\Delta x}{x}\right),$$

since the logarithm of the product of two quantities is equal to the sum of the logarithms of the quantities.

$$\therefore\ \Delta y=\log_\epsilon\left(1+\frac{\Delta x}{x}\right)=\frac{\Delta x}{x}-\tfrac{1}{2}\left(\frac{\Delta x}{x}\right)^2+\tfrac{1}{3}\left(\frac{\Delta x}{x}\right)^3-\ .\ .\ .\ +\ .\ .\ .$$

(Compare page 24.)

Diminishing Δx indefinitely, and neglecting terms involving the second and higher powers of Δx, we obtain

$$dy=\frac{dx}{x},$$

$$\frac{dy}{dx}=\frac{1}{x}.$$

It can easily be proved by the student that if

$$y=a\log_\epsilon x,$$

$$\frac{dy}{dx}=\frac{a}{x}.$$

Further, since $\log_b x=\dfrac{1}{\log_\epsilon b}\,.\,\log_\epsilon x=\log_b\epsilon\,.\,\log_\epsilon x$ (see page 25),

if

$$y=\mathsf{A}\log_b x,$$

$$\frac{dy}{dx}=\frac{\mathsf{A}}{\log_\epsilon b}\cdot\frac{1}{x}=\mathsf{A}\log_b\epsilon\,.\,\frac{1}{x}.$$

IV.

$$y=a\sin bx,$$

$$y+\Delta y=a\sin b(x+\Delta x)$$

$$=a(\sin bx\cos(b\Delta x)+\cos bx\sin(b\Delta x)).$$

(See Appendix, page 245.)

When Δx is made infinitely small (Δx being then written dx),

$$\cos(bdx)=1,$$

$$\sin(bdx)=bdx. \qquad \text{(See page 15.)}$$

$$\therefore\ y+dy=a(\sin bx\times 1+\cos bx\times bdx).$$

$$\therefore\ dy=ab\cos bx\cdot dx,$$

$$\frac{dy}{dx}=ab\cos bx.$$

V.

$$y=a\cos bx,$$

$$y+\Delta y=a\cos b(x+\Delta x)$$

$$=a(\cos bx\cos b\Delta x-\sin bx\sin b\Delta x).$$

When Δx is made infinitely small (Δx being then written dx),

$$\begin{aligned} y+dy &= a\cos bx - a\sin bx \times bdx, \\ dy &= -ab\sin bx \cdot dx, \\ \frac{dy}{dx} &= -ab\sin bx. \end{aligned}$$

VI.

$$\begin{aligned} y &= a\tan bx, \\ y+\Delta y &= a\tan b(x+\Delta x) \\ &= a\,\frac{\sin b(x+\Delta x)}{\cos b(x+\Delta x)} \\ &= a\,\frac{\sin bx\cos b\Delta x + \cos bx\sin b\Delta x}{\cos bx\cos b\Delta x - \sin bx\sin b\Delta x}. \end{aligned}$$

When Δx is made infinitely small (Δx being then written dx),

$$\begin{aligned} dy &= a\left\{\frac{(\sin bx\times 1)+(\cos bx\times bdx)}{(\cos bx\times 1)-(\sin bx\times bdx)} - \frac{\sin bx}{\cos bx}\right\} \\ &= a\,\frac{\sin bx\cos bx + \cos^2 bx\times bdx - \sin bx\cos bx + \sin^2 bx\times bdx}{\cos bx\,(\cos bx - \sin bx\cdot\times bdx)} \\ &= a\,\frac{(\cos^2 bx+\sin^2 bx)bdx}{\cos^2 bx - \cos bx\sin bx\times bdx}. \end{aligned}$$

Further, $\qquad \cos^2 bx + \sin^2 bx = 1.$

Also, the second term in the denominator of (7), being multiplied by dx, may be neglected in comparison with the first term. Hence

$$dy = ab\frac{dx}{\cos^2 bx},$$

$$\frac{dy}{dx} = \frac{ab}{\cos^2 bx}.$$

VII. $\qquad y = a\cot bx.$

By reasoning essentially similar to that employed in (VI.) it may be shown that

$$dy = -\frac{abdx}{\sin^2 bx},$$

$$\frac{dy}{dx} = -\frac{ab}{\sin^2 bx}.$$

The results obtained above may be conveniently collected for future reference.

I. If
$$y = ax^n,$$
then
$$dy = nax^{n-1}dx,$$
$$\frac{dy}{dx} = nax^{n-1}.$$

II. If
$$y = a^{bx},$$
then
$$dy = b \log_\epsilon a \cdot a^{bx}dx,$$
$$\frac{dy}{dx} = b \log_\epsilon a \cdot a^{bx}.$$

In particular, when
$$y = \mathsf{A}\epsilon^{bx},$$
$$dy = \mathsf{A}b\epsilon^{bx}dx,$$
$$\frac{dy}{dx} = \mathsf{A}b\epsilon^{bx}.$$

III. If
$$y = \mathsf{A} \log_b x,$$
then
$$dy = \frac{\mathsf{A}}{\log_\epsilon b} \frac{dx}{x} = \mathsf{A} \log_b \epsilon \cdot \frac{dx}{x},$$
$$\frac{dy}{dx} = \mathsf{A} \log_b \epsilon \cdot \frac{1}{x}.$$

In particular, when
$$y = \mathsf{A} \log_\epsilon x,$$
$$dy = \frac{\mathsf{A}}{x} dx,$$
$$\frac{dy}{dx} = \frac{\mathsf{A}}{x}.$$

IV. If
$$y = \mathsf{A} \sin bx,$$
then
$$dy = \mathsf{A}b \cos bx \cdot dx,$$
$$\frac{dy}{dx} = \mathsf{A}b \cos bx.$$

V. If
$$y = \mathsf{A} \cos bx,$$
then
$$dy = -\mathsf{A}b \sin bx \cdot dx,$$
$$\frac{dy}{dx} = -\mathsf{A}b \sin bx.$$

VI. If
$$y = \mathsf{A} \tan bx,$$
then
$$dy = \frac{\mathsf{A}b}{\cos^2 bx} dx,$$
$$\frac{dy}{dx} = \frac{\mathsf{A}b}{\cos^2 bx}.$$

VII. If
$$y = \mathsf{A} \cot bx,$$
then
$$dy = \frac{-\mathsf{A}b}{\sin^2 bx}\,dx,$$
$$\frac{dy}{dx} = \frac{-\mathsf{A}b}{\sin^2 bx}.$$

Exercises.—(1.) Find the inclination of the tangents to the curve $y = \tan x$, at the points where the curve cuts the axis of x.

Answer.—45°. (Compare Fig. 4.)

(2.) For what values of x are the tangents to the curves $y = \sin x$ and $y = \tan x$ coincident?

Answer.—$x = 0$, $x = 2\pi$, $x = 4\pi$, etc.

(3.) Show that the curves $y = \sin x$ and $y = \tan x$ intersect at right angles at $x = \pi$, $x = 3\pi$, $x = 5\pi$, etc.

(4.) Prove that, at points on a parabola at an infinite distance from the vertex, the tangents are perpendicular to the directrix.

(5.) Prove that a parabola is an ellipse with one focus at infinity.

Suggestion.—If the origin in Fig. 7 be shifted to the point Q_2, then for $x = \mathsf{OD}$ in Equation (1), page 19, we must substitute
$$\mathsf{OD} = \mathsf{Q}_2\mathsf{D} - \mathsf{Q}_2\mathsf{O} = x - a.$$

Let $\mathsf{Q}_2\mathsf{A} = \alpha$; then $a = \alpha + \delta$.
$$\frac{c^2}{4} = b^2 = a^2 - \delta^2 = (a - \delta)(a + \delta) = \alpha(\alpha + 2\delta).$$

Equation (1) now takes the form $\dfrac{\{x - (\alpha + \delta)\}^2}{(\alpha + \delta)^2} + \dfrac{y^2}{\alpha(\alpha + 2\delta)} = 1.$
$$\therefore \frac{x^2}{(\alpha + \delta)^2} - \frac{2x}{\alpha + \delta} + 1 + \frac{y^2}{\alpha(\alpha + 2\delta)} = 1.$$
$$\therefore y^2 - 2\alpha x\left(1 + \frac{\delta}{\alpha + \delta}\right) + x^2\alpha\left(\frac{1}{\alpha + \delta} + \frac{\delta}{(\alpha + \delta)^2}\right) = 0.$$

If the focus B is at infinity, then $\mathsf{AB} = 2\delta = \infty$. $\therefore \delta = \infty$.
$$\frac{1}{\alpha + \delta} = 0,\ \frac{\delta}{\alpha + \delta} = \frac{1}{\frac{\alpha}{\delta} + 1} = 1,\ \frac{\delta}{(\alpha + \delta)^2} = \frac{1}{\left(\frac{\alpha}{\delta} + 1\right)\left(\alpha + \delta\right)} = 0.$$

Hence the above equation reduces to
$$y^2 - 4\alpha x = 0.$$

CHAPTER III.

DIFFERENTIATION OF COMPLEX FUNCTIONS.

BEFORE proceeding to determine rules for the differentiation of compound expressions formed from combinations of the simple functions previously dealt with, it may prove serviceable to differentiate a few compound expressions from first principles :—

(1.) $y = x^n \sin ax.$

$$y + \Delta y = (x + \Delta x)^n \sin a(x + \Delta x)$$
$$= \left(x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{1 \cdot 2}x^{n-2}(\Delta x)^2 + \ . \ . \ . \ . \right)$$
$$\times (\sin ax \cos a\Delta x + \cos ax \sin a\Delta x).$$

Neglecting terms involving $(\Delta x)^2$, etc., in the first bracket, and writing $\cos a\Delta x = 1$, $\sin a\Delta x = a\Delta x$, in the second bracket, we obtain

$$y + \Delta y = (x^n + nx^{n-1}\Delta x)\ (\sin ax + \cos ax \cdot a\Delta x)$$
$$= x^n \sin ax + \Delta x\{nx^{n-1} \sin ax + ax^n \cos ax\}$$
$$+ anx^{n-1} \cos ax(\Delta x)^2.$$

Neglecting this last quantity, as involving $(\Delta x)^2$, we get

$$y + \Delta y = x^n \sin ax + \Delta x\{nx^{n-1} \sin ax + ax^n \cos ax\}.$$
$$\therefore\ \Delta y = \Delta x\{nx^{n-1} \sin ax + ax^n \cos ax\},$$
$$\frac{dy}{dx} = nx^{n-1} \sin ax + ax^n \cos ax.$$

(2.) $y = (1 - x^2)^{\frac{3}{2}}.$

$$y + \Delta y = \{1 - (x + \Delta x)^2\}^{\frac{3}{2}} = \{1 - x^2 - 2x\Delta x - (\Delta x)^2\}^{\frac{3}{2}}.$$

We may obviously neglect $(\Delta x)^2$ in comparison with the remaining terms. We then write

$$y+\Delta y=\{1-x^2-2x\Delta x\}^{\frac{3}{2}}=(1-x^2)^{\frac{3}{2}}\left\{1-\frac{2x\Delta x}{1-x^2}\right\}^{\frac{3}{2}}$$

$$=(1-x^2)^{\frac{3}{2}}\left\{1-\frac{3}{2}\cdot\frac{2x\Delta x}{1-x^2}+\text{terms involving }(\Delta x)^2,\text{etc.}\right\}.$$

$$\therefore\ y+\Delta y=(1-x^2)^{\frac{3}{2}}-\frac{3}{2}(1-x^2)^{\frac{1}{2}}\cdot 2x\Delta x+\ldots.$$

$$\therefore\ \Delta y=-3(1-x^2)^{\frac{1}{2}}x\Delta x,$$

$$\frac{dy}{dx}=-3(1-x^2)^{\frac{1}{2}}x.$$

(3.) $y=\tan^{-1}ax$.

The meaning of this equation must be carefully considered. Expressed in words, it denotes that y is an angle (measured, of course, in radians) such that its tangent is equal to ax. Hence,

$$ax=\tan y;\ \text{and}\ x=\frac{1}{a}\tan y.$$

From this it follows, in consonance with reasoning previously detailed, that

$$\frac{dx}{dy}=\frac{1}{a}\frac{1}{\cos^2 y};\ \therefore\frac{dy}{dx}=a\cos^2 y.$$

It is necessary, however, to express $\frac{dy}{dx}$ in terms of x, and not of y; that is, we must determine the value of $\cos^2 y$ in terms of x.

$$ax=\tan y=\frac{\sin y}{\cos y}=\frac{(1-\cos^2 y)^{\frac{1}{2}}}{\cos y}.$$

$$\therefore\ a^2x^2=\frac{1-\cos^2 y}{\cos^2 y},$$

$$\cos^2 y(a^2x^2+1)=1;\ \therefore\ \cos^2 y=\frac{1}{1+a^2x^2}.$$

Hence if $\qquad y=\tan^{-1}ax,$

$$\frac{dy}{dx}=a\cos^2 y=\frac{a}{1+a^2x^2}.$$

This is a most important result, and should be remembered by the student, as it often occurs in application of the higher mathematics to physics and mechanics.

The student will obtain a clearer insight into the working of the calculus by differentiating the following expressions from first principles, after the manner pursued in the above examples :—

(1.) $y = \sin^{-1} ax.$ $\quad \dfrac{dy}{dx} = \dfrac{a}{\sqrt{(1-a^2x^2)}}.$

(2.) $y = \cos^{-1} ax.$ $\quad \dfrac{dy}{dx} = -\dfrac{a}{\sqrt{1-a^2x^2}}.$

(3.) $y = \dfrac{a-x}{x}.$ $\quad \dfrac{dy}{dx} = -\dfrac{a}{x^2}.$

(4.) $y = x \log x.$ $\quad \dfrac{dy}{dx} = 1 + \log x.$

(5.) $y = \epsilon^x(1-x^3).$ $\quad \dfrac{dy}{dx} = \epsilon^x(1-3x^2-x^3).$

(6.) $y = (\sin x)^n \sin nx.$ $\quad \dfrac{dy}{dx} = n(\sin x)^{n-1} \sin (n+1)x.$

Differentiation of Complex Functions.

I.—To differentiate an expression composed of a sum of simple functions.

Let $\quad y = \mathsf{F}(x) + f(x) + \phi(x).$

Example :— $\quad y = ax + b \log_\epsilon x + \sin x.$

Let $\quad \mathsf{F}(x) = \xi, f(x) = \eta, \ \phi(x) = \zeta.$

$$y + \Delta y = (\xi + \Delta\xi) + (\eta + \Delta\eta) + (\zeta + \Delta\zeta).$$

$$\therefore \ \Delta y = \Delta\xi + \Delta\eta + \Delta\zeta,$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta\xi}{\Delta x} + \frac{\Delta\eta}{\Delta x} + \frac{\Delta\zeta}{\Delta x},$$

$$\frac{dy}{dx} = \frac{d\xi}{dx} + \frac{d\eta}{dx} + \frac{d\zeta}{dx}.$$

Hence the differential coefficient of the sum of a number of functions is equal to the sum of the differential coefficients of the constituent functions.

Examples :—

(1.) $$y = a + bx + cx^2 + dx^3,$$

$$\frac{dy}{dx} = b + 2cx + 3dx^2.$$

(2.)
$$y = ax + b \log_\epsilon x + \sin x,$$
$$\frac{dy}{dx} = a + \frac{b}{x} + \cos x.$$

II.—To differentiate a function of a function of x.

$$y = \mathsf{F}(f(x)).$$

The meaning of such an expression may be understood from a few examples. Thus, consider

$$y = \log(\sin x).$$

Sin x is a function of x—that is, a quantity whose value depends on the selected value of x. The logarithm of sin x will vary with the value of sin x, and hence with that of x.

Other examples :—

$$y = \sin(1 + x^2),$$
$$y = \log_\epsilon(x + \sqrt{x^2 - a^2}),$$
$$y = \mathsf{A}\epsilon^{\log_\epsilon x}, \qquad \text{etc., etc.}$$

Let $f(x) = \xi$.

Then
$$y = \mathsf{F}(f(x)) = \mathsf{F}(\xi),$$
$$y + \Delta y = \mathsf{F}(\xi + \Delta\xi), \qquad \text{etc. ; and finally,}$$
$$\frac{dy}{d\xi} = \frac{\mathsf{F}(\xi + d\xi) - \mathsf{F}(\xi)}{d\xi}.$$

Also,
$$\xi = f(x),$$
$$\frac{d\xi}{dx} = \frac{f(x + dx) - f(x)}{dx}.$$

It must be noticed that $d\xi$ in $\frac{d\xi}{dx}$ may obviously be supposed to have the same value as that of $d\xi$ in $\frac{dy}{d\xi}$. $d\xi$ is the difference in the value of ξ when x is increased to $x + dx$, and dy is the difference in the value of y when ξ is increased to $\xi + d\xi$. Hence it follows at once that

$$\frac{dy}{dx} = \frac{dy}{d\xi} \times \frac{d\xi}{dx},$$

since $d\xi$ may be cancelled in the two factors.

Hence the differential coefficient of y with respect to x is equal to the differential coefficient of y with respect to ξ, multiplied by the differential coefficient of ξ with respect to x.

Examples :—

(1.)
$$y=\sin(1+x^2),$$
$$\xi=(1+x^2),$$
$$y=\sin\,\xi,$$
$$\frac{dy}{d\xi}=\cos\,\xi,$$
$$\frac{d\xi}{dx}=2x.$$

(This may be found from first principles.)

$$\therefore\ \frac{dy}{dx}=\frac{dy}{d\xi}\times\frac{d\xi}{dx}=\cos\,\xi\cdot 2x=2x\,\cos(1+x^2).$$

(2.)
$$y=\mathsf{A}\epsilon^{\log_\epsilon x},$$
$$\xi=\log_\epsilon x.$$
$$y=\mathsf{A}\epsilon^{\xi},$$
$$\frac{dy}{d\xi}=\mathsf{A}\epsilon^{\xi},$$
$$\frac{d\xi}{dx}=\frac{1}{x}.$$
$$\therefore\ \frac{dy}{dx}=\frac{1}{x}\cdot\mathsf{A}\epsilon^{\xi}=\frac{\mathsf{A}}{x}\epsilon^{\log_\epsilon x}.$$

(3.)
$$y=\sqrt{a^2-x^2},$$
$$\xi=a^2-x^2,$$
$$y=\xi^{\frac{1}{2}},$$
$$\frac{dy}{d\xi}=\tfrac{1}{2}\xi^{-\frac{1}{2}},$$
$$\frac{d\xi}{dx}=-2x,$$
$$\frac{dy}{dx}=\frac{-2x}{2}\xi^{-\frac{1}{2}}=\frac{-x}{(a^2-x^2)^{\frac{1}{2}}}.$$

Similarly, if $y=(x^2-a^2)^{\frac{1}{2}}$, then $\dfrac{dy}{dx}=\dfrac{x}{(x^2-a^2)^{\frac{1}{2}}}$.

(4.)
$$y=\log_\epsilon(x+\sqrt{x^2-a^2}),$$
$$\xi=x+\sqrt{x^2-a^2}.$$

$$y = \log_\epsilon \xi,$$

$$\frac{dy}{d\xi} = \frac{1}{\xi},$$

$$\frac{d\xi}{dx} = 1 + \frac{x}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{(x^2 - a^2)^{\frac{1}{2}} + x}{(x^2 - a^2)^{\frac{1}{2}}}.$$

$$\frac{dy}{dx} = \frac{dy}{d\xi} \cdot \frac{d\xi}{dx} = \frac{1}{x + (x^2 - a^2)^{\frac{1}{2}}} \cdot \frac{(x^2 - a^2)^{\frac{1}{2}} + x}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{1}{(x^2 - a^2)^{\frac{1}{2}}}.$$

III.—To differentiate the product of two functions.

$$y = \mathsf{F}(x) f(x).$$

Example :—

$$y = x \sin x,$$

$$y = x \log x,$$

$$y = \log_\epsilon x \cdot \tan^{-1} x, \qquad \text{etc.}$$

Let $\mathsf{F}(x) = \xi, \quad f(x) = \zeta,$

$$y = \xi\zeta,$$

$$y + \Delta y = (\xi + \Delta\xi)(\zeta + d\zeta)$$

$$= \xi\zeta + \xi\Delta\zeta + \zeta\Delta\xi + (\Delta\xi\Delta\zeta).$$

The quantity between brackets in this last expression, being the product of two small quantities, may be neglected. Hence, remembering that

$$y = \xi\zeta,$$

$$\Delta y = \xi\Delta\zeta + \zeta\Delta\xi,$$

$$\frac{\Delta y}{\Delta x} = \xi\frac{\Delta\zeta}{\Delta x} + \zeta\frac{\Delta\xi}{\Delta x}.$$

$$\frac{dy}{dx} = \xi\frac{d\zeta}{dx} + \zeta\frac{d\xi}{dx}.$$

Hence, to differentiate the product of two functions, we in turn multiply each function by the differential coefficient of the other, and add the results.

Examples :—

(1.) $$y = x \sin x.$$

$$\xi = x, \quad \zeta = \sin x.$$

$$\frac{d\xi}{dx} = 1, \quad \frac{d\zeta}{dx} = \cos x.$$

$$\frac{dy}{dx} = (x \times \cos x) + (1 \times \sin x)$$
$$= x \cos x + \sin x.$$

(2.) $$y = x \log_\epsilon x.$$
$$\xi = x, \quad \zeta = \log_\epsilon x.$$
$$\frac{d\xi}{dx} = 1, \quad \frac{d\zeta}{dx} = \frac{1}{x}.$$
$$\frac{dy}{dx} = \left(x \times \frac{1}{x}\right) + (\log_\epsilon x \times 1)$$
$$= 1 + \log_\epsilon x.$$

(3.) $$y = \log_\epsilon x \tan^{-1} x.$$
$$\xi = \log_\epsilon x, \quad \zeta = \tan^{-1} x.$$
$$\frac{d\xi}{dx} = \frac{1}{x}, \quad \frac{d\zeta}{dx} = \frac{1}{1+x^2}.$$
$$\therefore \frac{dy}{dx} = \left(\log_\epsilon x \times \frac{1}{1+x^2}\right) + \left(\frac{1}{x} \times \tan^{-1} x\right)$$
$$= \frac{\log_\epsilon x}{1+x^2} + \frac{\tan^{-1} x}{x}.$$

IV.—To differentiate the quotient of two functions.

$$y = \frac{\mathsf{F}(x)}{f(x)}.$$

Example :— $$y = \frac{\log_\epsilon x}{x}, \quad y = \frac{\sin x}{\tan^{-1} x}, \qquad \text{etc.}$$

Let $$\mathsf{F}(x) = \xi, \; f(x) = \zeta,$$
$$y = \frac{\xi}{\zeta}.$$

When x is increased by Δx, let the respective increments of ξ, ζ, and y be $\Delta\xi$, $\Delta\zeta$, and Δy. Then

$$y + \Delta y = \frac{(\xi + \Delta\xi)}{(\zeta + \Delta\zeta)}.$$
$$\therefore \Delta y = \frac{\xi + \Delta\xi}{\zeta + \Delta\zeta} - y$$
$$= \frac{\xi + \Delta\xi}{\zeta + \Delta\zeta} - \frac{\xi}{\zeta}$$

$$=\frac{\zeta(\xi+\Delta\xi)-\xi(\zeta+\Delta\zeta)}{\zeta(\zeta+\Delta\zeta)}$$

$$=\frac{\zeta\Delta\xi-\xi\Delta\zeta}{\zeta^2+\zeta\Delta\zeta}.$$

In the denominator of this fraction, the quantity ζ^2, which is the square of a finite quantity, will be very large in comparison with $\zeta\Delta\zeta$, which is the product of two quantities, one of which, $\Delta\zeta$, is very small. Hence, neglecting $\zeta\Delta\zeta$ in comparison with ζ^2, we have

$$\Delta y=\frac{\zeta\Delta\xi-\xi\Delta\zeta}{\zeta^2}.$$

Divide both sides through by Δx. Then

$$\frac{\Delta y}{\Delta x}=\frac{\zeta\frac{\Delta\xi}{\Delta x}-\xi\frac{\Delta\zeta}{\Delta x}}{\zeta^2}.$$

When Δx and Δy, and also $\Delta\xi$ and $\Delta\zeta$, are diminished indefinitely, we have

$$\frac{dy}{dx}=\frac{\zeta\frac{d\xi}{dx}-\xi\frac{d\zeta}{dx}}{\zeta^2}.$$

Hence, to find the differential coefficient of the quotient of two functions, we subtract the product of the function in the numerator and the differential coefficient of the function in the denominator from the product of the function in the denominator and the differential coefficient of the function in the numerator, and divide the whole by the square of the function in the denominator.

Examples :—

(1.)
$$y=\frac{\log_e x}{x}.$$

$$\xi=\log_e x,\quad \zeta=x.$$

$$\frac{d\xi}{dx}=\frac{1}{x},\quad \frac{d\zeta}{dx}=1.$$

$$\therefore \frac{dy}{dx}=\frac{x\times\frac{1}{x}-\log_e x\times 1}{x^2}$$

$$=\frac{1-\log_e x}{x^2}.$$

(2.) $$y=\frac{\sin x}{\tan^{-1}x}.$$

$$\xi=\sin x,\quad \zeta=\tan^{-1}x.$$

$$\frac{d\xi}{dx}=\cos x,\quad \frac{d\zeta}{dx}=\frac{1}{(1+x)^2}.$$

$$\frac{dy}{dx}=\frac{\tan^{-1}x\cdot\times\cos x-\sin x\times\frac{1}{1+x^2}}{(\tan^{-1}x)^2}$$

$$=\frac{\cos x}{\tan^{-1}x}-\frac{\sin x}{(1+x^2)\,(\tan^{-1}x)^2}.$$

Success in differentiating complex functions will largely depend on the analytical skill possessed by the student. As far as the principles of the calculus are concerned, the foregoing rules will be found to cover the subject. On the other hand, problems may be proposed which require skill in algebraical or trigonometrical transformations. No general rules can be given for surmounting such difficulties, but the following problems may be taken as illustrating some of the most useful methods :—

(1.) $$y=(\sin x)^{\log_\epsilon x}.$$

Take logarithms of both sides of this equation.

$$\log_\epsilon y=\log_\epsilon x\times\log_\epsilon\sin x.$$

Since the differential coefficient of $\log_\epsilon x$ is $\frac{1}{x}$,

$$\frac{d\cdot(\log_\epsilon y)}{dx}=\frac{1}{y}\times\frac{dy}{dx},\text{ by Rule II.}$$

Then differentiating the quantity equated to $\log_\epsilon y$, and equating the result to the differential coefficient of $\log_\epsilon y$, we have—

$$\frac{1}{y}\frac{dy}{dx}=\frac{\log_\epsilon x\cdot\cos x}{\sin x}+\frac{\log_\epsilon\sin x}{x},\text{ by Rules II. and III.}$$

$$\therefore\ \frac{dy}{dx}=y\left(\log_\epsilon x\ \cot\cdot x+\frac{\log_\epsilon\sin x}{x}\right)$$

$$=(\sin x)^{\log_\epsilon x}\left(\log_\epsilon x\ \cot\cdot x+\frac{\log_\epsilon\sin x}{x}\right).$$

(2.) $$y=\sin\,(\log_\epsilon x).$$

Let $$\xi=\log_\epsilon x,\quad y=\sin\xi.$$

$$\frac{dy}{dx}=\frac{dy}{d\xi}\cdot\frac{d\xi}{dx}=\cos\xi\cdot\frac{1}{x}=\frac{\cos\,(\log_\epsilon x)}{x}.$$

(3.) $$y=\sin^{-1}\left(\frac{a-bx^2}{x^3}\right).$$

Let $$\xi=\frac{a-bx^2}{x^3}=\frac{a}{x^3}-\frac{b}{x},\quad y=\sin^{-1}\xi.$$

$$\frac{dy}{dx}=\frac{dy}{d\xi}\cdot\frac{d\xi}{dx}=\frac{1}{\sqrt{1-\xi^2}}\cdot\times\left(-\frac{3a}{x^4}+\frac{b}{x^2}\right)$$

$$=\frac{1}{\sqrt{1-\left(\frac{a-bx^2}{x^3}\right)^2}}\times\frac{bx^2-3a}{x^4}.$$

In the particular case where $a=4$, $b=3$, we have—

$$\frac{dy}{dx}=\frac{1}{\sqrt{\left(\frac{x^6-9x^4+24x^2-16}{x^6}\right)}}\times\frac{3(x^2-4)}{x^4}.$$

$$=\frac{x^3}{\sqrt{\{(x^2-1)(x^2-4)^2\}}}\times\frac{3(x^2-4)}{x^4}=\frac{3}{x\sqrt{(x^2-1)}}.$$

This example furnishes a good instance of the transformation which can be effected in a comparatively complicated result by the use of well-known algebraical devices.

Additional examples :—

(*a*) $y=\dfrac{1}{(x+\sqrt{1-x^2})}$. $\qquad \dfrac{dy}{dx}=\dfrac{x-\sqrt{1-x^2}}{\sqrt{(1-x^2)}\{x+\sqrt{(1-x^2)}\}^2}.$

(*b*) $y=\log\tan\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)$. $\qquad \dfrac{dy}{dx}=\dfrac{1}{\cos x}.$

(*c*) $y=\log_\epsilon\dfrac{(a+bx)}{a-bx}$. $\qquad \dfrac{dy}{dx}=\dfrac{2ab}{a^2-b^2x^2}.$

(Take $y=\log_\epsilon(a+bx)-\log_\epsilon(a-bx)$.)

(*d*) $y=\dfrac{\epsilon^x-\epsilon^{-x}}{\epsilon^x+\epsilon^{-x}}$. $\qquad \dfrac{dy}{dx}=\dfrac{4}{(\epsilon^x+\epsilon^{-x})^2}.$

(*e*) $y=\log(\epsilon^x+\epsilon^{-x})$. $\qquad \dfrac{dy}{dx}=\dfrac{\epsilon^x-\epsilon^{-x}}{\epsilon^x+\epsilon^{-x}}.$

(*f*) $y=\log_\epsilon(x+\sqrt{x^2+a^2})$. $\qquad \dfrac{dy}{dx}=\dfrac{1}{(x^2+a^2)^{\frac{1}{2}}}.$

CHAPTER IV.

MAXIMA AND MINIMA.

REMEMBERING the definition which has already been given of a function, it will readily be seen that the curve ABCDEFG (Fig. 14) may be represented analytically by the equation

$$y = \mathsf{F}(x).$$

The above curve is characterized by the occurrence of certain points, B, D, F, which possess ordinates greater than those of the

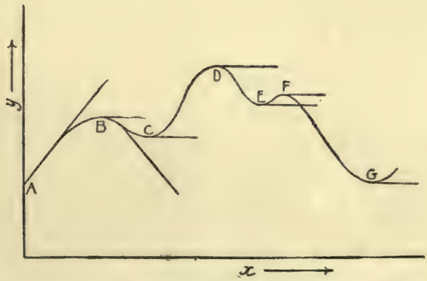

FIG. 14.

points in their immediate vicinity. For instance, proceeding along the curve from A, it is obvious that the magnitude of the y ordinate increases until the point B is reached, and immediately afterwards commences to decrease. The points B, D, F of the curve are said to be maximum points, or at these points y is said to possess maximum values.

The term *maximum* must therefore be understood, not in the absolute sense (in which D alone would possess a maximum value), but as referring to a small part only of the curve. The essential for a maximum point is, that the value of y increases up to the point in question, and then decreases.

Consequently, a curve may possess any number of maximum points. Some curves, such as those representing sine and cosine functions, possess an infinite number of maximum points.

For similar reasons, the points C, E, G are termed *minimum* points. In approaching one of these points the value of y diminishes, whilst after passing through that point the value of y increases.

If the curve ABCDEFG is represented by the equation

$$y = \mathsf{F}(x),$$

then $\frac{dy}{dx}$ will represent the trigonometrical tangent of the angle which the geometrical tangent to the curve at the point (x, y) makes with the axis of x.

It can easily be seen that the value of $\frac{dy}{dx}$ will vary from point to point on the curve. The important point to notice is, *that at maximum and minimum points the tangent is parallel to the x axis;* in other words, at these points

$$\frac{dy}{dx} = 0.$$

This gives us a ready means of determining the co-ordinates of points possessing maximum or minimum values. We simply express y in terms of x, obtain the value of $\frac{dy}{dx}$, and equate this to zero. We thus obtain an equation in x, which, when solved, gives us the abscissæ of the various maximum and minimum points of the curve.

The mathematical method of determining whether a particular value of x corresponds to a maximum or to a minimum point on the curve will be discussed subsequently. In a great many problems it is quite easy, from general reasoning, or from the nature of the problem, to discriminate between these two cases.

A few simple problems will now be discussed.

Problem.—A circle of radius r is drawn, and it is required to draw two radii in such positions that the area of the triangle formed by these radii, and a straight line joining their points of intersection with the circumference of the circle, shall possess a maximum value.

Let ABD (Fig. 15) be the circle, and AC, BC any two radii. It is required to draw AC, BC in such relative positions that the area of the triangle ABC shall have a maximum value.

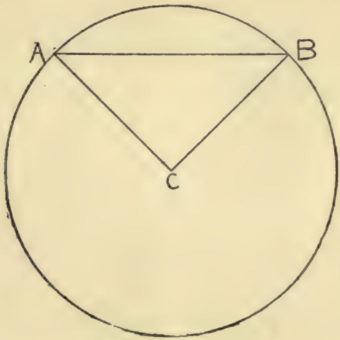

FIG. 15.

If two radii are drawn in a straight line, thus constituting a diameter of the circle (the angle ACB then being equal to 180°), the area of the triangle will possess a zero value.

If the angle between the two radii is diminished in successive stages, the area of the triangle will at first increase. When, however, the angle $ACB = 0°$, the area of the triangle will once more possess zero value. Hence, whilst increasing the angle ACB from 0° to 180°, the area of the triangle at first increases, reaches a maximum value, and then diminishes.

It is required to find the value of the angle ACB, for which the triangle possesses a maximum value.

Let angle $ACB = x$.

Let area of triangle $ACB = y$.

Let radius of circle $= r$.

Then, since the area of a triangle is equal to half the product of the adjacent sides into the sine of the included angle, we have

$$\text{Area of triangle } ACB = y = \tfrac{1}{2}r^2 \sin x.$$

When y has a maximum value, $\dfrac{dy}{dx}$ will be equal to zero.

$$y = \tfrac{1}{2}r^2 \sin x.$$

$$\frac{dy}{dx} = \tfrac{1}{2}r^2 \cos x = 0, \text{ for maximum value of } y$$

$$\therefore \cos x = 0.$$

$$\therefore x = \frac{\pi}{2}.$$

Therefore the radii must be drawn at right angles to each other.

Fig. 16 roughly represents the manner in which the area of the

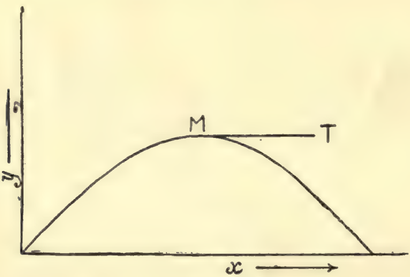

FIG. 16.

triangle varies as the angle ACB is increased from 0 to π. The point M is the required maximum point, and here the tangent MT is horizontal, and $\frac{dy}{dx} = 0$.

Problem.—It is required to cut the largest possible rectangular strip from a circular piece of paper.

Let ABCD (Fig. 17) be a rectangle enclosed in a circle. It is required that the sides AB, BC shall be so proportioned that the area ABCD shall be a maximum.

Let O be the centre of the circle, and let OB, the radius of the circle, be represented by r.

Draw OE perpendicular to the line BC.

Let the angle BOE $= x$.
Area of rectangle $= y$.
Then BC $= 2$BE $= 2r \sin x$.
Also, AB $=$ FE $= 2$OE $= 2r \cos x$.

$\therefore$ Area of rectangle ABCD $= y =$ AB $\times$ BC $= 4r^2 \sin x \cos x$.

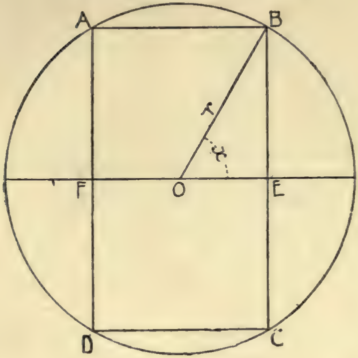

FIG. 17.

Remembering that $2 \sin x \cos x = \sin 2x$, we have—

$$y = 2r^2 \sin 2x.$$

$$\therefore \frac{dy}{dx} = 4r^2 \cos 2x.$$

For y to be a maximum, $\frac{dy}{dx} = 0$.

$$\therefore \cos 2x = 0,$$

$$2x = \frac{\pi}{2},$$

$$x = \frac{\pi}{4} = 45^\circ.$$

Therefore, $\quad \mathsf{BC} = 2r \sin 45^\circ = 2r \times \frac{1}{\sqrt{2}},$

$$\mathsf{AB} = 2r \cos 45^\circ = 2r \times \frac{1}{\sqrt{2}} = \mathsf{BC}.$$

Therefore the largest rectangle which can be cut from a circular piece of paper is a square, of which the four corners lie on the circumference of the circle.

Problem.—It is required to draw the ellipse which, with a given perimeter, shall possess a maximum area.

Given that (*a*) *perimeter of ellipse** $= \pi (x + z)$, where x and z are the principal semi-axes of the ellipse; and (*b*) *area of ellipse* $= \pi xz$.

* This is only a rough approximation holding when the axes are not far from equal. The accurate length of the perimeter of an ellipse can only be expressed as a series.

It can easily be seen that an ellipse possessing semi-axes, of which one is very long and the other correspondingly short, will possess an extremely small area.

If such an ellipse is taken, and the longer axis is diminished by successive small stages, the shorter axis being correspondingly enlarged, then the area will increase. At a certain stage a circle will be formed, the two semi-axes then being equal. A further diminution of the same axis will produce an ellipse with its major axis at right angles to the major axis of the original ellipse. Hence the area will finally diminish, and the maximum value of the area will occur when the area ceases to increase, and is just about to diminish.

Let $p =$ the given perimeter of the ellipse. Then, since p is to possess an invariable value,

$$p = \pi(x + z).$$

$$\therefore \ z = \frac{p}{\pi} - x.$$

Let $y =$ area of ellipse.

Then
$$y = \pi xz = \pi x\left(\frac{p}{\pi} - x\right)$$
$$= px - \pi x^2,$$
$$\frac{dy}{dx} = p - 2\pi x.$$

For y to be a maximum,

$$\frac{dy}{dx} = p - 2\pi x = 0.$$

$$\therefore \ x = \frac{p}{2\pi}.$$

To find z, we have

$$z = \frac{p}{\pi} - x = \frac{p}{\pi} - \frac{p}{2\pi} = \frac{p}{2\pi};$$

that is, the two principal axes of the ellipse are equal, and the particular ellipse is therefore a circle.

The above property of an ellipse, of possessing, when the semi-axes are equal, a maximum area for a given perimeter, has been utilized by Professor Boys in designing leaden water-pipes which will not burst during frosty weather. The pipes are made of elliptical sections, and the expansion of the water on freezing simply tends to make the section more circular, since this will produce a greater sectional area, and therefore a greater volume for a given length of pipe. Such pipes were frozen many times in succession without a rupture occurring, whilst pipes of circular section were invariably burst at the second or third freezing.

Problem.—A given point P (Fig. 18) is situated in the quadrant between the axes of x and y. It is required to draw a straight line through P, cutting the axes in the points A and B, subject to the condition that the length AB shall be as small as possible.

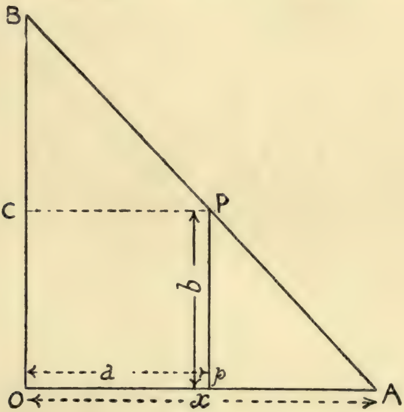

FIG. 18.

We will solve the above problem in two different ways, in order to exemplify the variety in the methods by which such problems may be attacked.

Since the position of the point P is given, its co-ordinates, $\mathrm{O}p = a$, $p\mathrm{P} = b$, must be known. Let $\mathrm{OA} = x$. Then, since the triangles $\mathrm{AP}p$, ABO are similar, we have

$$\frac{\mathrm{BO}}{\mathrm{OA}} = \frac{\mathrm{P}p}{p\mathrm{A}}. \qquad \therefore \frac{\mathrm{BO}}{x} = \frac{b}{x-a}. \qquad \therefore \mathrm{BO} = \frac{bx}{x-a}.$$

Let the length of the line $\mathrm{AB} = u$.

$$\text{Then } u = \sqrt{(\mathrm{OA})^2 + (\mathrm{BO})^2} = \sqrt{x^2 + \left(\frac{bx}{x-a}\right)^2}.$$

It is required to find the value of x, which will give to $u=\left(x^2+\frac{b^2x^2}{(x-a)^2}\right)^{\frac{1}{2}}$ a minimum value.

First of all, it can easily be seen that u may acquire a minimum value. For if $\mathsf{OA}=x$ is increased indefinitely, the line BA will become more and more nearly parallel to the axis of x, and will therefore be indefinitely increased in length. Starting with the line BA very nearly parallel to OA, it can be seen that as the angle BAO is increased, the length of BA will at first decrease. When, however, the angle BAO approximates to a right angle, the line BA will be approximately parallel to the axis OB, and will therefore again have acquired an indefinitely great length. Hence, as the point A is moved toward p from an indefinitely great distance along OA, the length of the line BA will at first decrease, and subsequently increase. Therefore there must be some position of the point A, defined by a value of x between ∞ and a, for which the line BA possesses a minimum length.

$$u=\left(x^2+\frac{b^2x^2}{(x-a)^2}\right)^{\frac{1}{2}}.$$

$$\frac{du}{dx}=\frac{1}{2}\left(x^2+\frac{b^2x^2}{(x-a)^2}\right)^{-\frac{1}{2}}\times\left\{2x+\frac{2b^2x(x-a)^2-2(x-a)b^2x^2}{(x-a)^4}\right\}$$

$$=\frac{1}{2}\left(x^2+\frac{b^2x^2}{(x-a)^2}\right)^{-\frac{1}{2}}\times 2x\left\{\frac{(x-a)^4+b^2(x-a)^2-b^2x(x-a)}{(x-a)^4}\right\}.$$

$$\therefore\ \frac{du}{dx}=\frac{x\,\dfrac{(x-a)^4+b^2(x-a)^2-b^2x(x-a)}{(x-a)^4}}{\dfrac{(x^2(x-a)^2+b^2x^2)^{\frac{1}{2}}}{x-a}}$$

$$=\quad x\,\frac{\dfrac{(x-a)^3+b^2(x-a)-b^2x}{(x-a)^2}}{(x^2(x-a)^2+b^2x^2)^{\frac{1}{2}}}.$$

For u to be a minimum, $\frac{du}{dx}=0$.

$$\therefore\ x\frac{\dfrac{(x-a)^3+b^2(x-a)-b^2x}{(x-a)^2}}{(x^2(x-a)^2+b^2x^2)^{\frac{1}{2}}}=0\ . \qquad . \qquad (1).$$

This equation now requires careful consideration. If $x=0$, the value of the expression on the left-hand side will be reduced to zero, since it is multiplied by x. The solution so obtained does not, however, apply to the problem in hand, since, as previously explained, x varies only from $\propto$ to a.

We can therefore divide the equation through by x.

We can multiply the whole expression by the quantity in the denominator—namely,

$$(x^2(x-a)^2+b^2x^2)^{\frac{1}{2}},$$

and so remove this quantity from the equation, provided we are sure that it will not become equal to zero or infinity for any value of x contemplated in the problem.

The quantity in question would become equal to ∞ if $x=\infty$ —a value excluded from the problem.

The expression would become equal to zero if $x=0$—another value excluded from the problem.

Squaring, and dividing through by x^2, we get—

$$(x-a)^2+b^2.$$

This cannot be equal to $\propto$ unless $x=\propto$. For it to be equal to 0,

$$x=a\pm\sqrt{-b^2},$$

which is an impossible solution, since it involves the square root of a negative quantity.

We therefore conclude that we may multiply (1) by the quantity in the denominator.

Performing this operation, we obtain

$$\frac{(x-a)^3+b^2(x-a)-b^2x}{(x-a)^2}=0.$$

We may multiply this through by $(x-a)^2$, since, when $x=a$ (and the denominator consequently becomes equal to zero), the line BA will be parallel to OB, which position cannot obviously correspond to a minimum value for the length of OA.

Finally we have—

$$(x-a)^3+b^2(x-a)-b^2x=(x-a)^3-b^2a=0.$$

$$\therefore\ x=a+a^{\frac{1}{3}}b^{\frac{2}{3}}.$$

This value of x will make the length of the line AB a minimum. To find this minimum length, we have—

$$\mathsf{BA}=\sqrt{x^2+\frac{b^2x^2}{(x-a)^2}}=\frac{x\sqrt{(x-a)^2+b^2}}{x-a}$$
$$=\frac{\left(a+a^{\frac{1}{3}}b^{\frac{2}{3}}\right)\left\{a^{\frac{2}{3}}b^{\frac{4}{3}}+b^2\right\}^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{2}{3}}}.$$

Dividing the quantity between the first brackets by $a^{\frac{1}{3}}b^{\frac{2}{3}}$, and removing the quantity $b^{\frac{4}{3}}$ from the second brackets, we get for the length of BA—

$$u=\left(a^{\frac{2}{3}}b^{-\frac{2}{3}}+1\right)\times b^{\frac{2}{3}}\times\left\{a^{\frac{2}{3}}+b^{\frac{2}{3}}\right\}^{\frac{1}{2}}$$
$$=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)\left\{a^{\frac{2}{3}}+b^{\frac{2}{3}}\right\}^{\frac{1}{2}}=\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

As solved above, the problem in hand requires a certain amount of analytical skill to obtain the correct solution. It has been worked out in this manner in order to point out a number of considerations which are frequently useful. A much simpler solution will now be given.

Instead of taking x as an independent variable, let us take the angle $\mathsf{BAO}=\theta$. Draw a line PC through P, perpendicular to OB.

Then $\qquad \dfrac{b}{\mathsf{PA}}=\sin\theta. \qquad \therefore\ \mathsf{PA}=\dfrac{b}{\sin\theta}.$

Similarly, $\qquad \dfrac{\mathsf{PC}}{\mathsf{PB}}=\dfrac{p\mathsf{O}}{\mathsf{PB}}=\dfrac{a}{\mathsf{PB}}=\cos\theta.$

$$\therefore\ \mathsf{PB}=\frac{a}{\cos\theta}.$$

Then length $\mathsf{BA}=u=\mathsf{PB}+\mathsf{PA}=\dfrac{a}{\cos\theta}+\dfrac{b}{\sin\theta}.$

$$\therefore\ \frac{du}{d\theta}=+\frac{a\sin\theta}{\cos^2\theta}-\frac{b\cos\theta}{\sin^2\theta}.$$

When $\dfrac{du}{d\theta}=0,$

$$\frac{a\sin\theta}{\cos^2\theta}-\frac{b\cos\theta}{\sin^2\theta}=0 \quad . \quad . \quad . \quad (2).$$

$$\therefore\ \frac{\sin^3\theta}{\cos^3\theta}=\tan^3\theta=\frac{b}{a} \quad . \quad . \quad . \quad (3).$$

$$\therefore\ \tan\theta=\sqrt[3]{\frac{b}{a}}.$$

To find the point A, we have

$$\frac{b}{p\mathrm{A}} = \tan\,\theta = \sqrt[3]{\frac{b}{a}}.$$

$$\therefore\ p\mathrm{A} = \frac{a^{\frac{1}{3}}}{b^{\frac{1}{3}}} \times b = a^{\frac{1}{3}} b^{\frac{2}{3}}.$$

Since $p\mathrm{A} = x - a$, we have

$$x = a + a^{\frac{1}{3}} b^{\frac{2}{3}},$$

the result previously obtained.

It should be noticed that in simplifying (2) we multiply by $\frac{\sin^2\theta}{\cos\,\theta}$. This is permissible, since this quantity can be equal to neither 0 nor ∞ when the line AB has a minimum value.

From the above it will be seen that by a suitable choice of co-ordinates much analytical labour may be avoided. For this purpose no special rules can be given, but the student is advised to obtain the solution of a few problems, using different variables.

Problem.—A person in a boat 3 miles from the nearest point of the beach (which is supposed to be straight), wishes to reach a place 5 miles from that point along the beach in the shortest possible time. Supposing he can row at the rate of 4 miles per hour, and walk at the rate of 5 miles per hour, at what point on the beach should he land?

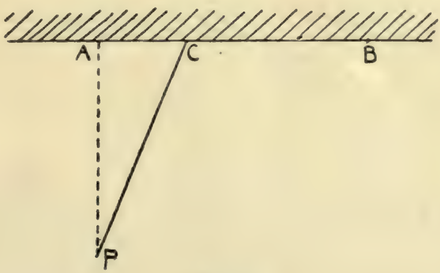

FIG. 19.

Let P (Fig. 19) be the initial position of the boat, at a distance PA = 3 miles from land; and let B be the point, 5 miles from A (that is, AB = 5), which he wishes to reach. It is plain that he should row to some point, C, between A and B, and then walk from C to B, if he wishes to reach B in the shortest possible time.

Let t be the time required to reach B by the route PCB.

Let distance $CB = x$.

Then distance $AC = (5 - x)$ miles.

$$\text{Distance } PC = \{3^2 + (5 - x)^2\}^{\frac{1}{2}} = (9 + 25 - 10x + x^2)^{\frac{1}{2}}$$
$$= (34 - 10x + x^2)^{\frac{1}{2}}.$$

Time required to walk distance $CB = \dfrac{x}{5}$.

Time required to row distance $PC = \dfrac{(34 - 10x + x^2)^{\frac{1}{2}}}{4}$.

Time required to reach B by route PCB

$$= t = \frac{x}{5} + \frac{(34 - 10x + x^2)^{\frac{1}{2}}}{4}.$$

For t to be a minimum, $\dfrac{dt}{dx} = 0$.

$$\therefore \frac{dt}{dx} = \frac{1}{5} + \left(\frac{1}{2} \times \frac{1}{4} \times \frac{2x - 10}{(34 - 10x + x^2)^{\frac{1}{2}}}\right) = 0.$$

$$\therefore 8(34 - 10x + x^2)^{\frac{1}{2}} = 5(10 - 2x).$$

Squaring, we get

$$64(34 - 10x + x^2) = 25(100 - 40x + 4x^2) = 2500 - 1000x + 100x^2.$$

Collecting similar terms, we get

$$36x^2 - 360x + 324 = 0.$$

Dividing through by 36, we get

$$x^2 - 10x + 9 = 0.$$

Factorizing, we finally obtain

$$(x - 1)(x - 9) = 0.$$

The solutions are—

$$x = 1, \text{ and } x = 9.$$

A glance at Fig. 18 will show that $x = 9$ cannot be a solution of the problem, since the point C would then be to the left of A. Hence, finally, $x = 1$ gives the required result—that is, the man must row directly to a point 1 mile from his destination, and walk the remainder of the way.

The student may be left to prove that the result obtained is a minimum and not a maximum value of t.

Example.—Find the condition that, in the figure given on page 59, the sum of the distances OA and OB should possess the smallest possible value.

Ans. Distance $OA = a + \sqrt{ba}$,

$$\tan \theta = \sqrt{\frac{b}{a}}.$$

Problem.—It is required to cut a beam of rectangular section from a tree-trunk of circular section, subject to the condition that the beam shall be the stiffest possible under the circumstances. Determine the dimensions of the section of the beam.

The stiffness of a beam varies inversely as the depression produced at the middle of the beam by a weight of 1 lb. placed there. In other words, if a beam which is straight when supported near its ends (Fig. 20) takes up the position indicated by the dotted

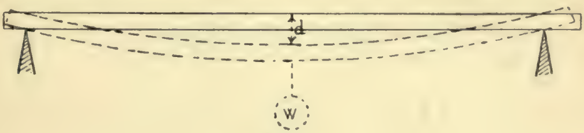

FIG. 20.

lines when a load of w lbs. is suspended from its middle point, the depression at the centre being d, then stiffness of beam $= 1$ $\div$ depression of middle point per lb. suspended $= \frac{1}{\frac{d}{w}} = \frac{w}{d}$.

In order to obtain a beam of maximum stiffness, we must arrange that d has the smallest possible value, or that d should be a minimum.

It can be proved, and will be proved in these pages later on, that d varies inversely as the product of the breadth into the cube of the depth of the beam.

Let r be the radius of the circular section of the tree-trunk.

Let $B =$ breadth of the beam.

Let $D =$ depth of the beam.

EFGH (Fig. 21) represents the required section of the beam.

We have $d \propto \frac{1}{\text{BD}^3}$.

d must possess a minimum value, and therefore BD^3 must possess a maximum value.

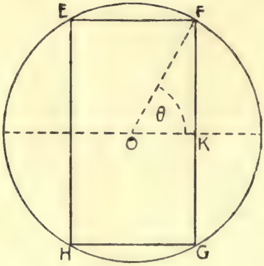

Fig. 21.

Let O (Fig. 21) be the centre of the circular section of the trunk, and let OK be drawn perpendicular to the side FG of the rectangle EFGH. Draw the radius OF, and let the angle $\text{FOK} = \theta$.

$$\text{Then FG} = \text{D} = 2 \times \text{OF} \sin \theta = 2r \sin \theta.$$

$$\text{EF} = \text{B} = 2 \times \text{OF} \cos \theta = 2r \cos \theta.$$

$$\therefore\ 2r \cos \theta \times (2r \sin \theta)^3 \text{ must be a maximum.}$$

$$\therefore\ 16r^4 \sin^3 \theta \cos \theta = \text{maximum.}$$

From the nature of the problem r must be constant.
Let $y = \sin^3 \theta \cos \theta$. Then, for a maximum value of y,

$$\frac{dy}{d\theta} = 3 \sin^2 \theta \cos^2 \theta - \sin^4 \theta = 0.$$

$$\sin^2 \theta = 3 \cos^2 \theta.$$

$$\therefore\ \sin \theta = \sqrt{3} \cos \theta,$$

$$\tan \theta = \sqrt{3},$$

$$\text{and } 2r \sin \theta = \sqrt{3} \times 2r \cos \theta.$$

$$\text{D} = \sqrt{3} \times \text{B}.$$

$$\text{Since } \tan \theta = \sqrt{3},\ \theta = 60°.$$

The above conditions are sufficient to determine the required dimensions of the section.

It may easily be seen that the result obtained applies to a maximum and not a minimum value of y.

For when $\frac{dy}{d\theta} = 0$, BD^3 must possess either a maximum or minimum value.

As θ is taken more and more nearly equal to 0, the value of D decreases indefinitely, whilst B approaches the limiting value r.

$$\text{Therefore for } \theta = 0,$$
$$BD^3 = 0.$$

Also, when $\theta = \frac{\pi}{2}$, $D = r$, $B = 0$. Therefore for this value of θ also, $BD^3 = 0$.

BD^3 can never possess a negative value, from the nature of the problem. Consequently, as

$$BD^3 = 0 \text{ when } \theta = 0,$$
$$BD^3 = 0 \text{ when } \theta = \frac{\pi}{2},$$
$$BD^3 = \text{positive value for intermediate values of } \theta.$$

BD^3 must at first increase in value as θ is increased from zero upwards, and must subsequently decrease as the value $\theta = \frac{\pi}{2}$ is approached. Hence there must be a maximum value of BD^3 between the values $\theta = 0$ and $\theta = \frac{\pi}{2}$. This maximum value is given, as shown above, by the condition $\theta = 60°$.

Exercise.—The equation to a certain curve is $y = \epsilon^{-ax} \sin bx$. Show that maximum and minimum values of y correspond to values of x determined from the equation

$$\tan bx = \frac{b}{a}.$$

Trace the general form of the curve, and compare it with that given by the equation

$$y = \sin bx.$$

CHAPTER V.

EXPANSION OF FUNCTIONS.

WE have already had occasion to use the binomial theorem, especially in relation to the expansion

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{1 \cdot 2} x^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots .$$

In this expansion a certain function of x is developed in the form of a series of ascending powers of x. If x is less than unity, successive terms rapidly diminish in value, and the series is said to be convergent. We can then substitute the value of x in the first two or three terms, neglecting the remaining terms, and so find the value of $(1 \pm x)^n$. We have already used this expansion for theoretical purposes (see page 35); we shall here give examples showing its applicability to practical computations.

Example.—Find the value of $\sqrt[3]{976}$.

$$976 = 1000 - 24.$$

$$\therefore \sqrt[3]{976} = (1000 - 24)^{\frac{1}{3}} = \sqrt[3]{1000} \cdot \left(1 - \frac{24}{1000}\right)^{\frac{1}{3}} = 10(1 - \cdot 024)^{\frac{1}{3}}.$$

By the binomial theorem,

$$(1 - \cdot 024)^{\frac{1}{3}} = 1 - \tfrac{1}{3} \times \cdot 024 + \frac{\frac{1}{3}(\frac{1}{3} - 1)}{1 \cdot 2} \cdot (\cdot 024)^2 - \ldots .$$

$$= 1 - \cdot 008 - \cdot 00006 - \ldots .$$

The third term is so small as to be practically negligible. Hence we have, finally,

$$\sqrt[3]{976} = 10(1 - \cdot 008) = 9 \cdot 92.$$

Example.—Find the value of $\frac{1}{958}$ correct to four significant figures.

$$958 = 1000 - 42.$$

$$\therefore \frac{1}{958} = \frac{1}{1000(1 - \cdot 042)} = \frac{1}{1000}(1 - \cdot 042)^{-1}.$$

By the binomial theorem,

$$(1 - \cdot 042)^{-1} = 1 + \cdot 042 + \frac{(-1)(-2)}{1 \cdot 2}(\cdot 042)^2 + \ldots .$$

$$= 1 + \cdot 042 + \cdot 001764 + \ldots .$$

The inclusion of the third term only affects the third decimal figure, and the fourth term would only affect the fourth decimal figure—that is, the fifth significant figure. Hence the fourth and all succeeding terms in the expansion may be neglected.

$\therefore (1 - \cdot 042)^{-1} = 1 \cdot 044$, correct to the fourth significant figure.

$$\therefore \frac{1}{958} = \frac{1}{1000}(1 - \cdot 042)^{-1} = \cdot 001044.$$

These two examples will serve to show the usefulness of the above theorem.

There are a great many functions which cannot be expanded by the aid of the binomial theorem. We shall now develop another method by which these functions can be expanded in a series of ascending powers of x.

Any series so obtained will only be valid when it proves to be convergent—that is, when the values of successive terms diminish rapidly. In more advanced works on the calculus, methods are developed by which the convergence or divergence of a series may be easily determined. In the present case these will be omitted, and, as a consequence, the applicability of any particular series must be ascertained during the computation which it is intended to facilitate.

Successive Differentiation.—We have learned to give a definite meaning to the expression

$$\frac{dy}{dx},$$

in which y is supposed to be a function of x. Thus, if

$$y = a \sin x,$$
$$\frac{dy}{dx} = a \cos x.$$

The result obtained is termed the first differential coefficient of y with respect to x. This quantity will itself generally be a function of x, and is therefore capable of being differentiated. In this case, since we differentiate $\frac{dy}{dx}$ instead of y, we symbolize the operation by

$$\frac{d\frac{dy}{dx}}{dx} \text{ or } \frac{d^2y}{dx^2}.$$

In the first of these expressions, $\frac{dy}{dx}$ is substituted for y in the symbol for the first differential coefficient of y with respect to x. The second expression is the usual way of writing the result, which is termed the second differential coefficient of y with respect to x.

The student should carefully distinguish between the expressions

$$\frac{d^2y}{dx^2} \text{ and } \left(\frac{dy}{dx}\right)^2.$$

The first of these has already been defined. The second expression stands for the square of the first differential coefficient of y with respect to x.

Thus, if

$$y = a \sin x,$$
$$\frac{dy}{dx} = a \cos x,$$
$$\left(\frac{dy}{dx}\right)^2 = a^2 \cos^2 x,$$
$$\frac{d^2y}{dx^2} = \frac{d(a \cos x)}{dx} = -a \sin x.$$

We can in turn differentiate the second differential coefficient of y with respect to x, the result being termed the third differential coefficient, and written

$$\frac{d^3y}{dx^3}.$$

Similarly, the fourth, fifth, sixth, etc., differential coefficients can be obtained, the nth differential coefficient being written

$$\frac{d^ny}{dx^n}.$$

Another way of indicating these operations is as follows :—

$$\begin{aligned} \text{If} \quad & y = f(x), \\ & \frac{dy}{dx} \text{ is written } f'(x), \\ & \frac{d^2y}{dx^2} \quad \text{,,} \quad \text{,,} \quad f''(x), \\ & \frac{d^3y}{dx^3} \quad \text{,,} \quad \text{,,} \quad f'''(x), \end{aligned}$$

the number of dashes placed above at the right-hand side of the f serving to indicate the order of the differentiation.

Examples.—Find the values of $\frac{dy}{dx}$, $\left(\frac{dy}{dx}\right)^2$, $\frac{d^2y}{dx^2}$, $\left(\frac{d^2y}{dx^2}\right)^2$, and $\frac{d^3y}{dx^3}$, when y has the following values :—

(1.) $y = a \tan bx$.
(2.) $y = 1 - x^2$.
(3.) $y = a \cos bx$.
(4.) $y = a \log_e \cdot x$.
(5.) $y = a\epsilon^{bx}$.

Answers.—

(1.) $\frac{ab}{\cos^2 bx}$, $\frac{a^2b^2}{\cos^4 bx}$, $\frac{2ab^2 \sin bx}{\cos^3 bx}$, $\frac{4a^2b^4 \sin^2 bx}{\cos^6 bx}$, $\frac{2ab^3 \cos^2 bx + 6ab^3 \sin^2 x}{\cos^4 bx}$.

(2.) $-2x$, $4x^2$, -2, 4, 0.

(3.) $-ab \sin bx$, $a^2b^2 \sin^2 bx$, $-ab^2 \cos bx$, $a^2b^4 \cos^2 bx$, $ab^3 \sin bx$.

(4.) $\frac{a}{x}$, $\frac{a^2}{x^2}$, $\frac{-a}{x^2}$, $\frac{a^2}{x^4}$, $\frac{2a}{x^3}$

(5.) $ab\epsilon^{bx}$, $a^2b^2\epsilon^{2bx}$, $ab^2\epsilon^{bx}$, $a^2b^4\epsilon^{2bx}$, $ab^3\epsilon^{bx}$

Problem.—Let it be required to expand $f(x)$, a given function of x, in a series of ascending powers of x; in other words, to determine the value of the coefficients A, B, C, D . . . . in the identity

$$f(x) = \mathsf{A} + \mathsf{B}x + \mathsf{C}x^2 + \mathsf{D}x^3 + \mathsf{E}x^4 + \dots \quad (1).$$

Let $y = f(x)$. Then this result is capable of being expressed graphically as a curve, in which we suppose each value of x, treated as an abscissa, to correspond to a single valued ordinate y. If the equation (1) is true for all values of x, then on substituting any particular value of x in the series to the right of (1)—this series being assumed to be convergent—we shall obtain the same value for y as if the same value of x had been substituted directly in $f(x)$.

Since (1) is true for all values of x, it must be true for the particular value $x = 0$. Substitute this value of x in (1), and let the resulting value of $f(x)$ be denoted by $f(x)_0$. Then we have

$$f(x)_0 = \mathsf{A} + \mathsf{B} \times 0 + \mathsf{C} \times 0^2 + \mathsf{D} \times 0^3 + \mathsf{E} \times 0^4 + \dots$$
$$\therefore \mathsf{A} = f(x)_0 \quad (2).$$

Further, the two expressions

$$y = f(x)$$

and

$$y = \mathsf{A} + \mathsf{B}x + \mathsf{C}x^2 + \mathsf{D}x^3 + \mathsf{E}x^4 + \dots$$

must represent the same curve. Consequently, since $\frac{dy}{dx}$ represents the tangent of the angle of slope of the curve at the point (x, y), $\frac{dy}{dx}$ must have the same value, whether $f(x)$, or the series to which it is equal, is differentiated. The first differential coefficient of $f(x)$ may be written, as stated above, as $f'(x)$. Hence, equating this quantity to the first differential coefficient of the series—which can be found by combining rule 1, chap. ii., with rule 1, chap. iii. (see pages 35 and 45)—we find that

$$f'(x) = \mathsf{B} + 2\mathsf{C}x + 3\mathsf{D}x^2 + 4\mathsf{E}x^3 + \dots \quad (3).$$

This latter equation must be true for all values of x, and therefore for the particular value $x = 0$. Let $f'(x)_0$ be used to denote the value of the first differential coefficient of $f(x)$, when x is put equal to 0. Then we have

$$f'(x)_0 = \mathsf{B} + 2\mathsf{C} \times 0 + 3\mathsf{D} \times 0^2 + 4\mathsf{E} \times 0^3.$$
$$\therefore \mathsf{B} = f'(x)_0 \quad (4).$$

Reasoning similar to that already employed shows that we may differentiate both sides of (3) and equate the results to each other. Therefore

$$f''(x) = 2\mathsf{C} + 2\cdot3\cdot\mathsf{D}x + 3\cdot4\cdot\mathsf{E}x^2 + \ldots \qquad (5).$$

This result must be true for all values of x, and therefore for $x=0$. Substituting this value, we find that

$$2\mathsf{C} = f''(x)_0, \ \therefore \ \mathsf{C} = \tfrac{1}{2}f''(x)_0 \ldots \qquad (6).$$

Differentiating both sides of (5), and equating the results to each other, we get

$$f'''(x) = 2\cdot3\,\mathsf{D} + 2\cdot3\cdot4\cdot\mathsf{E}x + \ldots \qquad (7).$$

Hence, as before,

$$\mathsf{D} = \frac{1}{2\cdot3\cdot}f'''(x)_0 \qquad (8).$$

Differentiating both sides of (7), and equating the results to each other, we get

$$f''''(x) = 2\cdot3\cdot4\cdot\mathsf{E} + \ldots \qquad (9).$$

$$\therefore \ \mathsf{E} = \frac{1}{2\cdot3\cdot4}f''''(x)_0 \qquad (10).$$

As a result, we find from (2), (4), (6), (8), (10),

$$\mathsf{A} = f(x)_0.$$

$$\mathsf{B} = \frac{1}{1}f'(x)_0.$$

$$\mathsf{C} = \frac{1}{1\cdot2}f''(x)_0.$$

$$\mathsf{D} = \frac{1}{1\cdot2\cdot3}f'''(x)_0.$$

$$\mathsf{E} = \frac{1}{1\cdot2\cdot3\cdot4}f''''(x)_0.$$

Etc.

Substituting these values in (1), we finally obtain the result that

$$f(x) = f(x)_0 + \frac{x}{1}f'(x)_0 + \frac{x^2}{1\cdot2}f''(x)_0 + \frac{x^3}{1\cdot2\cdot3}f'''(x)_0 + \frac{x^4}{1\cdot2\cdot3\cdot4}f''''(x)_0 + \ldots \qquad (11),$$

in which $f(x)$ has been expanded in a series of ascending powers of x, with constant coefficients.

This result is known as Maclaurin's Theorem.

Example 1.—Expand $\sin x$ in a series of ascending powers of x.

$$\begin{aligned}
f(x) &= \sin x. & \therefore\ f(x)_0 &= \sin 0 = 0.\\
f'(x) &= \cos x. & \therefore\ f'(x)_0 &= \cos 0 = 1.\\
f''(x) &= -\sin x. & \therefore\ f''(x)_0 &= -\sin 0 = -0.\\
f'''(x) &= -\cos x. & \therefore\ f'''(x)_0 &= -\cos 0 = -1.\\
f''''(x) &= \sin x. & \therefore\ f''''(x)_0 &= \sin 0 = 0.\\
f'''''(x) &= \cos x. & \therefore\ f'''''(x)_0 &= \cos 0 = 1.\\
&\cdots & &\cdots
\end{aligned}$$

Hence, substituting these values in (11) above, we find that

$$\sin x = x - \frac{x^3}{1\cdot2\cdot3} + \frac{x^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{x^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \cdots \quad (12).$$

This result is one of great importance, both from a theoretical and from a practical point of view. In the first place, when x (which, it must be remembered, represents the circular measure of an angle) is small, x^3, x^5, etc., may be neglected in comparison with the first term—namely, x. Hence, for small values of x,

$$\sin x = x,$$

a result with which we are already acquainted. (See page 15.)

As long as x is less than 1, the series (12) will obviously be convergent, and we may use it to compute the value of $\sin x$.

Example 2.—Compute, by the aid of Maclaurin's Theorem, the value of the sine of 25°.

We must first express 25° in circular measure.

Since the circular measure of

$$180° = \pi = 3{\cdot}14159,$$

$$25° = \frac{25}{180} \times 3{\cdot}14159 = {\cdot}43633.$$

$$\begin{aligned}
\therefore\ \sin 25° = \sin {\cdot}43633 &= {\cdot}43633 - \frac{({\cdot}43633)^3}{1\cdot2\cdot3} + \frac{({\cdot}43633)^5}{1\cdot2\cdot3\cdot4\cdot5} - \cdots\\
&= {\cdot}43633 - {\cdot}01384 + {\cdot}00013 - \cdots\\
&= {\cdot}42262.
\end{aligned}$$

Example 3.—Expand cos x in a series of ascending powers of x.

Here
$$\begin{aligned} f(x) &= \cos x. & \therefore\ f(x)_0 &= \cos 0 = 1. \\ f'(x) &= -\sin x. & \therefore\ f'(x)_0 &= -\sin 0 = 0. \\ f''(x) &= -\cos x. & \therefore\ f''(x)_0 &= -\cos 0 = -1. \\ f'''(x) &= \sin x. & \therefore\ f'''(x)_0 &= \sin 0 = 0. \\ f''''(x) &= \cos x. & \therefore\ f''''(x)_0 &= \cos 0 = 1. \\ \cdot\ \cdot &= \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \end{aligned}$$

$$\therefore\ \cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \dots.$$

Example 4.—Compute, by the aid of Maclaurin's Theorem, the value of the cosine of 25°.

$$\begin{aligned} \cos 25^\circ = \cos \cdot 43633 &= 1 - \cdot 09519 + \cdot 00150 - \cdot 00001 \dots. \\ &= \cdot 90630. \end{aligned}$$

Example 5.—Expand a^x in a series of ascending powers of x.

$$\begin{aligned} f(x) &= a^x. & \therefore\ f(x)_0 &= a^0 = 1. \\ f'(x) &= \log_\epsilon a\cdot a^x. & \therefore\ f'(x)_0 &= \log_\epsilon a. \\ f''(x) &= (\log_\epsilon a)^2 a^x. & \therefore\ f''(x)_0 &= (\log_\epsilon a)^2. \\ f'''(x) &= (\log_\epsilon a)^3 a^x. & \therefore\ f'''(x)_0 &= (\log_\epsilon a)^3. \end{aligned}$$

$$\therefore\ a^x = 1 + \log_\epsilon a\cdot x + \frac{(\log_\epsilon a)^2}{1\cdot 2}x^2 + \frac{(\log_\epsilon a)^3}{1\cdot 2\cdot 3} + \dots.$$

This is the result already obtained on page 24. The above cannot, however, be considered to give an independent proof of the results previously obtained, since the expansion of a^x in a series of ascending powers of x was used in differentiating a^x (see page 38), and hence the agreement exhibited above only serves as a check on the accuracy of Maclaurin's Theorem.

Example 6.—Expand $\log_\epsilon(1+x)$ in a series of ascending powers of x.

$$\begin{aligned} f(x) &= \log_\epsilon(1+x). & \therefore\ f(x)_0 &= \log_\epsilon 1 = 0. \\ f'(x) &= \frac{1}{1+x}. & \therefore\ f'(x)_0 &= 1. \\ f''(x) &= \frac{-1}{(1+x)^2}. & \therefore\ f''(x)_0 &= -1. \\ f'''(x) &= \frac{(-1)(-2)}{(1+x)^3}. & \therefore\ f'''(x)_0 &= +2. \end{aligned}$$

$$f''''(x) = \frac{(-1)(-2)(-3)}{(1+x)^4}. \quad \therefore f''''(x)_0 = -(1\cdot 2\cdot 3).$$

$$\cdot \quad \cdot \quad = \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\therefore \log_\epsilon(1+x) = 0 + \frac{x}{1} + \frac{-1x^2}{1\cdot 2} + \frac{+2x^3}{1\cdot 2\cdot 3} + \frac{-(1\cdot 2\cdot 3)x^4}{1\cdot 2\cdot 3\cdot 4} \cdot \cdot$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdot \cdot \cdot \cdot$$

(Compare Equation 8, page 25.)

We may proceed in a similar manner to expand $\log_\epsilon(1-x)$ in a series of ascending powers of x. A simple method, however, is to substitute $-x$ for x in the series just obtained. We then get

$$\log_\epsilon(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdot \cdot \cdot \cdot$$

$$\therefore \log_\epsilon \frac{1+x}{1-x} = \log_\epsilon(1+x) - \log_\epsilon(1-x)$$

$$= 2\left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \cdot \cdot \cdot \cdot \right\}.$$

Real and Imaginary Quantities.—The square root of a positive number is itself a number which may be positive or negative, and we may easily assign a definite geometrical signification to this result. Thus, if the area of a square is equal to a^2, the length of a side of the square may be represented by $\pm a$, the sign prefixed denoting the direction in which the length a is measured. $\pm a$ is said to be a real quantity.

In the course of mathematical investigations we often obtain results involving the square root of a minus quantity. Now there is no *number* which, when multiplied by itself, will give a minus value; hence the square root of a minus quantity cannot represent a number. A very important signification can, however, be given to the expressions of the nature of $\sqrt{-a^2}$, or its equivalent, $\sqrt{-1}\cdot a$.

We have already found it convenient to agree that certain symbols should denote operations. Thus, the expression

$$\sin x$$

means that we are to draw two lines meeting at an angle whose circular measure is x, and then from a point on one line to draw a perpendicular to the other, and obtain the ratio

$$\frac{\text{perpendicular}}{\text{hypotenuse}}$$

in the right-angled triangle so formed. Similarly, the symbol $\frac{d}{dx}$, written before y, means that we are to perform on y the operation known as differentiation. Hence $\frac{d}{dx}$, and the expression *sin*, etc., may be called *operators*—that is, symbols denoting the performance of definite operations.

Now let us suppose that we require an operator which shall rotate a line through a definite angle, about an axis at right angles to the line. Let AB (Fig. 22) be the line in its initial position, the positive direction being from A to B. Let it be required to find an operator which shall rotate this line, about an axis through A, into the position AC.

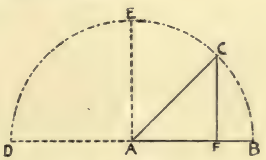

Fig. 22.

If the angle CAB were 180°—that is, if the line were to be rotated about the point A till it occupied the position AD—the necessary operator would obviously be -1; since the length AD is equal to AB, but is measured in the opposite direction.

Further, if the length AB $= a$, and if ι be used to indicate an operator which will rotate the line AB into the position AE, the line AE may be represented, both in magnitude and direction, by ιa. In other words, multiplying a length a by the operator ι rotates that length through 90°, in the opposite sense to that in which the hands of a clock revolve. If we apply the operator ι to the length AE (that is, ιa), we shall rotate it to the position AD.

Therefore $$\iota \times \iota a = \iota^2 a = -a.$$
$$\therefore \iota^2 = -1,$$
$$\iota = \sqrt{-1}.$$

Hence, distances measured parallel to AB may be denoted by letters, and those at right angles to AB by letters preceded by the operator $\sqrt{-1}$, or ι. The quantity multiplied by ι or $\sqrt{-1}$ is termed an *imaginary quantity*; and an expression of the form

$$\alpha + \sqrt{-1}\beta,$$

which consists of a real and imaginary part, is termed a *complex quantity.*

Let us assume that the line AB may be rotated into the position AC by being multiplied by a complex quantity. For example, we will examine the case where $\angle$ CAB $= 45^\circ$.

Then the line AC may be represented, *in magnitude and direction*, by

$$(\alpha + \iota\beta)a,$$

if the line AB is represented, in magnitude and direction, by the symbol a.

Apply the operator $(\alpha + \iota\beta)$, which rotates a line through an angle of 45° in a sense opposite to that in which the hands of a clock revolve, to the line AC. We shall thus rotate the line into the position AE, which we have already agreed to indicate by ιa.

$$\therefore\ (\alpha + \iota\beta)\mathrm{AC} = (\alpha + \iota\beta)\,(\alpha + \iota\beta)a = (\alpha + \iota\beta)^2 a = \iota a.$$

But $$(\alpha + \iota\beta)^2 = \alpha^2 + 2\iota\alpha\beta + \iota^2\beta^2 = \alpha^2 - \beta^2 + 2\iota\alpha\beta,$$

since $$\iota^2 = -1.$$

$$\therefore\ \alpha^2 - \beta^2 + 2\iota\alpha\beta = \iota.$$

On the left-hand side of this last expression we have a real quantity, $\alpha^2 - \beta^2$, added to an imaginary quantity, $2\iota\alpha\beta$, and the sum of these is equal to a quantity wholly imaginary—that is, ι. Hence the real quantity $\alpha^2 - \beta^2$ must be equal to 0, and $2\alpha\beta$ must be equal to the coefficient of ι on the right-hand side of the equation—that is, to 1.

$$\alpha^2 - \beta^2 = 0. \quad \therefore\ \alpha = \pm\beta.$$

$$2\alpha\beta = 1. \quad \therefore\ \alpha = \beta = \frac{1}{\sqrt{2}}\cdot$$

Therefore the line AC, making an angle of 45° with AB, may be represented, in magnitude and direction, by

$$\left(\frac{1}{\sqrt{2}} + \iota\frac{1}{\sqrt{2}}\right)a.$$

But $$\mathrm{AF} = \frac{a}{\sqrt{2}} = a\cos 45^\circ,$$

$$\mathrm{FC} = \frac{a}{\sqrt{2}} = a\sin 45^\circ.$$

$$\therefore\ \overrightarrow{\mathrm{AC}} = a(\cos 45^\circ + \iota\sin 45^\circ).$$

In a similar manner it may be shown that if the angle $\mathsf{CAB} = x$, the line AC may be represented, in magnitude and direction, by

$$\overrightarrow{\mathsf{AC}} = a(\cos x + \iota \sin x).$$

Looking at the problem from a slightly different point of view, if $\overrightarrow{\mathsf{AC}}$ represents a displacement in magnitude and direction, this is equivalent to the sum of the component displacements $\overrightarrow{\mathsf{AF}}$ and $\overrightarrow{\mathsf{FC}}$; or

$$\overrightarrow{\mathsf{AC}} = \overrightarrow{\mathsf{AF}} + \overrightarrow{\mathsf{FC}} = a \cos x + \iota a \sin x.$$

We have already expressed $\cos x$ and $\sin x$ in the form of an infinite series of ascending powers of x. Hence,

$$\cos x + \iota \sin x = 1 + \iota x - \frac{x^2}{\underline{|2}} - \iota\frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \iota\frac{x^5}{\underline{|5}} - \ldots .$$

On page 24 it is proved that

$$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \quad \ldots$$

Substituting ιx for x in this expression, and remembering that $\iota^2 = -1$, $\iota^3 = -\iota$, $\iota^4 = (-1)^2 = 1$, etc., we get

$$\begin{aligned}\epsilon^{\iota x} &= 1 + \iota x + \frac{\iota^2 x^2}{\underline{|2}} + \frac{\iota^3 x^3}{\underline{|3}} + \frac{\iota^4 x^4}{\underline{|4}} \\ &= 1 + \iota x - \frac{x^2}{\underline{|2}} - \iota\frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \ldots . \\ &= \cos x + \iota \sin x.\end{aligned}$$

Hence the operator $\epsilon^{\iota x}$ rotates a straight line, in a sense opposite to that in which the hands of a clock revolve, through an angle whose circular measure is x.

If a radius, AC (Fig. 22), is supposed to rotate from its initial position AB, at a uniform rate, about its extremity A, a time T being required for a complete rotation, then the angle swept out in a time t will be

$$\frac{2\pi t}{\mathsf{T}},$$

and the position of the radius at the time t may be represented by the expression

$$a\epsilon^{\iota\frac{2\pi t}{T}}.$$

Substituting $-\iota x$ for x in the expansion for ϵ^x, we get

$$\epsilon^{-\iota x}=1+\frac{-\iota x}{1}+\frac{(-\iota x)^2}{\lfloor 2}+\frac{(-\iota x)^3}{\lfloor 3}+\frac{(-\iota x)^4}{\lfloor 4}+\ldots,$$

$$=1-\frac{\iota x}{1}-\frac{x^2}{\lfloor 2}+\iota\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}-\ldots,$$

$$=\cos x-\iota\sin x.$$

$$\therefore \epsilon^{\iota x}=\cos x+\iota\sin x,$$

$$\epsilon^{-\iota x}=\cos x-\iota\sin x.$$

$$\therefore \cos x=\frac{\epsilon^{\iota x}+\epsilon^{-\iota x}}{2},$$

$$\sin x=\frac{\epsilon^{\iota x}-\epsilon^{-\iota x}}{2\iota}.$$

The above are termed the exponential values of the sine and cosine.

Taylor's Theorem.—It is often necessary, having given the value of a function for a particular value of x, to find the value of the same function for a slightly different value of x.

For instance, given that $\sin 30^\circ = \cdot 5$, it may be required to calculate the value of $\sin 31^\circ$ correct to four places of decimals. The theorem about to be discussed may be advantageously used for this purpose.

We must first prove the following proposition :—Let $f(x)$ be a given function of x, and let $f(x+\xi)$ be the value of this function when $x+\xi$ is substituted for x throughout. (For instance, if $f(x)=\sin x$, then $f(x+\xi)=\sin(x+\xi)$. If $f(x)=a+bx+cx^2$, then $f(x+\xi)=a+b(x+\xi)+c(x+\xi)^2$.) Then the result obtained when $f(x+\xi)$ is differentiated with respect to x, treating ξ as a constant, is the same as that obtained when $f(x+\xi)$ is differentiated with respect to ξ, treating x as a constant.

The student should satisfy himself that the above statement is true, with respect to the functions given above, by performing the differentiations. A general proof will here be given :—

Let $\quad x+\xi=z. \quad \therefore \frac{dz}{dx}=1$, and $\frac{dz}{d\xi}=1.$

Then $$\frac{df(x+\xi)}{dx} = \frac{df(z)}{dx} = \frac{df(z)}{dz}\,\frac{dz}{dx} = \frac{df(z)}{dz} = f'(z).$$ (See p. 71.)

$$\frac{df(x+\xi)}{d\xi} = \frac{df(z)}{d\xi} = \frac{df(z)}{dz}\,\frac{dz}{d\xi} = \frac{df(z)}{dz} = f'(z).$$

We can now proceed to expand $f(x+\xi)$ in a series of ascending powers of ξ, with coefficients independent of ξ.

Let $f(x+\xi) = \mathsf{A} + \mathsf{B}\xi + \mathsf{C}\xi^2 + \mathsf{D}\xi^3 + \mathsf{E}\xi^4 + \ldots \ldots$ (1),

where A, B, C, D, E . . . . do not contain ξ.

Put $\xi = 0$ in 1. Then

$$f(x) = \mathsf{A} \quad \ldots \ldots \quad (2).$$

Differentiate both sides of (1) with respect to ξ.

$$\therefore f'(x+\xi) = \mathsf{B} + 2\mathsf{C}\xi + 3\mathsf{D}\xi^2 + 4\mathsf{E}\xi^3 + \ldots \ldots \quad (3).$$

Put $\xi = 0$ in (3). Then

$$f'(x) = \mathsf{B} \quad \ldots \ldots \quad (4).$$

Differentiate (3) throughout with respect to ξ. Then

$$f''(x+\xi) = 2\mathsf{C} + 2\cdot 3\mathsf{D}\xi + 3\cdot 4\xi^2 + \ldots \ldots \quad (5).$$

Put $\xi = 0$ in (5). Then

$$f''(x) = 2\mathsf{C}. \quad \therefore \mathsf{C} = \tfrac{1}{2}f''(x) \quad \ldots \quad (6).$$

Differentiate (5) throughout with respect to ξ. Then

$$f'''(x+\xi) = 2\cdot 3\cdot \mathsf{D} + 2\cdot 3\cdot 4\mathsf{D}\xi + \ldots \ldots \quad (7).$$

Put $\xi = 0$ in (7). Then

$$f'''(x) = 2\cdot 3\cdot \mathsf{D}. \quad \therefore \mathsf{D} = \frac{1}{2\cdot 3}f'''(x) \quad \ldots \quad (8).$$

Proceeding in a similar manner, it may be shown that

$$\mathsf{E} = \frac{1}{2\cdot 3\cdot 4}f''''(x), \text{ etc.} \quad \ldots \quad (9).$$

Substituting from (2), (4), (6), (8), (9), in 1, we get

$$f(x+\xi) = f(x) + \frac{f'(x)}{\underline{|1}}\xi + \frac{f''(x)}{\underline{|2}}\xi^2 + \frac{f'''(x)}{\underline{|3}}\xi^3 + \frac{f''''(x)}{\underline{|4}}\xi^4 + \ldots \quad (10).$$

Example.—Compute the value of sin 31° = sin (30° + 1°).

Here ξ is equal to the circular measure equivalent to 1°—that is, to $\dfrac{3{\cdot}14159}{180} = {\cdot}017453$.

$$f(x) = \sin 30^\circ = {\cdot}5.$$

$$f'(x) = \frac{d}{dx} \cdot \sin x = \cos x = \cos 30^\circ = \frac{\sqrt{3}}{2} = {\cdot}8660.$$

$$f''(x) = \frac{d}{dx} \cdot \cos x = -\sin x = -\sin 30^\circ = -{\cdot}5.$$

$$f'''(x) = \frac{d}{dx} \cdot (-\sin x) = -\cos x = -\cos 30^\circ = -{\cdot}8660.$$

$$\ldots\ldots = \ldots\ldots$$

$\therefore$ from (10),

$$\sin 31^\circ = \sin 30^\circ + \frac{\cos 30^\circ}{\underline{|1}} \cdot {\cdot}017453 - \frac{\sin 30^\circ}{\underline{|2}} \cdot ({\cdot}017453)^2 - \frac{\cos 30^\circ}{\underline{|3}} \cdot ({\cdot}017453)^3 + \ldots\ldots$$

$$= {\cdot}5 + {\cdot}01511 - {\cdot}00007 - \ldots\ldots$$

$$= {\cdot}51504.$$

In this example the terms rapidly diminish in value, so that the first three terms may be taken as equivalent in value (to the degree of accuracy specified) to the whole series.

Application to Maxima and Minima Problems.—Let $y = f(x)$ be the equation to a curve, such as that given in Fig. 14, page 53. We have already considered the definition given of maximum and minimum points on the curve. For $f(x)$ to be a maximum, $f(x - \Delta x)$ and $f(x + \Delta x)$ must both be less than $f(x)$; or, in other words, y must increase up to the point corresponding to the abscissa x, and must then decrease.

But, from Taylor's Theorem,

$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1} \Delta x + \frac{f''(x)}{\underline{|2}} (\Delta x)^2 + \ldots\ldots$$

$$f(x - \Delta x) = f(x) + \frac{f'(x)}{1} \cdot (-\Delta x) + \frac{f\ (x)}{\underline{|2}} (-\Delta x)^2 + \ldots\ldots$$

If Δx is very small, terms involving $(\Delta x)^2$, etc., will be very small in comparison with that involving (Δx). Therefore, to a first approximation, we may write—

$$f(x+\Delta x)=f(x)+f'(x)\Delta x \quad . \quad . \quad . \quad (1).$$
$$f(x-\Delta x)=f(x)-f'(x)\Delta x \quad . \quad . \quad . \quad (2).$$

But if the right-hand side of (1) is greater than $f(x)$, the right-hand side of (2) must be less than $f(x)$, unless $f'(x)$ is equal to zero. In other words, in passing from the point corresponding to the abscissa $(x-\Delta x)$, through that corresponding to x, to that corresponding to $(x+\Delta x)$, y must continually increase, or continually diminish, unless $f'(x)=0$. But, for a maximum value, y must increase in passing from $(x-\Delta x)$ to x, and then diminish in passing to $(x+\Delta x)$. For a minimum value, y must diminish in passing from $(x-\Delta x)$ to x, and then increase in passing to $(x+\Delta x)$. Hence, for a maximum or minimum value, $f'(x)$ must be equal to zero, or

$$\frac{dy}{dx}=0.$$

This is the result already obtained.

When $\qquad f'(x)=0,$

$$f(x+\Delta x)=f(x)+\frac{f''(x)}{\lfloor 2}(\Delta x)^2 \quad . \quad (3).$$

$$f(x-\Delta x)=f(x)+\frac{f''(x)}{\lfloor 2}(-\Delta x)^2$$

$$=f(x)+\frac{f''(x)}{\lfloor 2}(\Delta x)^2 \quad . \quad . \quad (4).$$

If $f''(x)$ or $\frac{d^2y}{dx^2}$ is positive, the value of the right-hand side of both (3) and (4) will be greater than $f(x)$; in other words, y will diminish in passing from the point corresponding to $(x-\Delta x)$ to that corresponding to x, and will subsequently increase. In this case y has a *minimum value.*

If $f''(x)$ or $\frac{d^2y}{dx^2}$ is negative, it can be seen, by the aid of reasoning similar to that just used, that y is a maximum.

Hence, to determine whether a particular value of x, *determined from the equation*

$$f'(x)=\frac{dy}{dx}=0,$$

corresponds to a maximum or minimum value of y, find the value of $\frac{d^2y}{dx^2}$, *and substitute in this the value of x in question. If the result is a positive quantity, x corresponds to a minimum value of y. If the result is negative, x corresponds to a maximum value of y.*

Exercise.—Find the values of x which give maximum and minimum values to the expression $x^3 - 9x^2 + 15x - 6$, and discriminate between them.

Let $y = x^3 - 9x^2 + 15x - 6$.

For y to be a maximum or minimum, $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 3x^2 - 18x + 15 = 0 \quad . \quad . \quad (1).$$

$$x^2 - 6x + 5 = 0.$$

$$\therefore (x-3)^2 = 9 - 5 = 4.$$

$$x - 3 = \pm 4.$$

$$x = 3 \pm 4 = 7 \text{ or } -1.$$

Differentiating (1) above, we find

$$\frac{d^2y}{dx^2} = 6x - 18.$$

When $x = 7$,

$$\frac{d^2y}{dx^2} = 42 - 18 = 34, \text{ a positive number.}$$

$\therefore x = 7$ corresponds to a minimum value of y.

When $x = -1$,

$$\frac{d^2y}{dx^2} = -6 - 18 = -24, \text{ a negative number.}$$

$\therefore x = -1$ corresponds to a maximum value of y.

Problem.—To determine the radius of curvature at a point (x, y) on a curve represented by the equation $y = f(x)$.

The meaning to be attached to the term "radius of curvature" may be understood from the following :—

If three points in a plane are given, then one, and only one, circle can be drawn through these points. As a consequence, if three points, B, C, D (Fig. 23), be taken on the curve ABCDE, then a circle can be drawn through these three points. As the points B, C, D are taken closer and closer together, the portion of the circle between B and D will more and more nearly coincide with the portion of the curve ABCDE which lies between the same points. The centre, K, of the circle drawn through three successive points, B, C, D, indefinitely near to each other, is termed the *centre of curvature* of the curve at B. The radius KB of the circle is termed the *radius of curvature* of the curve at B.

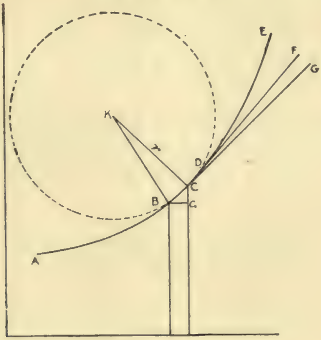

FIG. 23.

Draw a straight line, BCG, through the first two points. Then, when A and B are taken indefinitely near to each other, this line will be the tangent to the curve at B. Similarly, if the straight line CDF is drawn through the points C and D, this will, in the limit, be the tangent to the curve at C. Join KB, KC. Then KB will be perpendicular to BCG, and KC will be perpendicular to CDF. Hence the angle BKC will be equal to the angle FCG.

Let KB, the radius of curvature at B, be equal to R, and let the angle BKC be equal to $d\theta$.

Then the infinitesimal arc $\mathrm{BC} = \mathrm{R}d\theta$.

But if x, y are the co-ordinates of B, and $x+dx$, $y+dy$ are the co-ordinates of C,

$$\text{Arc}\quad \mathrm{BC} = \sqrt{\mathrm{B}c + c\mathrm{C}} = \{(dx)^2 + (dy)^2\}^{\frac{1}{2}} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$

$$\therefore\ \mathrm{R}d\theta = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx \quad . \quad . \quad . \quad . \quad (1).$$

It remains to determine $d\theta$ in terms of dy and dx, etc.

Let $\frac{dy}{dx} = \phi(x) =$ the tangent of the angle of inclination of the geometrical tangent BCG at the point B on the curve.

Then, since the abscissa of C is $(x + dx)$, the tangent of the angle of inclination of the geometrical tangent CDE, at the point C on the curve, will be equal to $\phi(x + dx)$.

But by Taylor's Theorem,

$$\phi(x + dx) = \phi(x) + dx \cdot \phi'(x) + \ldots .$$

Also, if θ be the inclination of the tangent at B to the axis of x, $(\theta + d\theta)$ will be the inclination of the tangent at the point C. Here $d\theta = \angle$ FCG $= \angle$ CKB, as already explained.

By Taylor's Theorem,

$$\tan(\theta + d\theta) = \tan\theta + d\theta \cdot \frac{d}{d\theta}\tan\theta + \ldots .$$

$$= \tan\theta + d\theta \cdot \frac{1}{\cos^2\theta} + \ldots .$$

$$\therefore \phi(x) + dx \cdot \phi'(x) = \tan\theta + \frac{d\theta}{\cos^2\theta} \text{ in the limit.}$$

Since $\phi(x) = \tan\theta$, we have

$$\phi'(x)dx = \frac{d\theta}{\cos^2\theta} \cdot$$

Now $\phi(x) = \frac{dy}{dx} \cdot \quad \therefore \phi'(x) = \frac{d^2y}{dx^2} \cdot$

$$\therefore \frac{d^2y}{dx^2}dx = \frac{d\theta}{\cos^2\theta} \cdot$$

$$\therefore d\theta = \cos^2\theta \times \frac{d^2y}{dx^2} \cdot dx.$$

But $\tan\theta = \frac{dy}{dx} \cdot$

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \cos^2\theta = \frac{1}{1 + \left(\frac{dy}{dx}\right)^2}.$$

$$\therefore d\theta = \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} dx.$$

Substituting this value for $d\theta$ in (1), page 85, we get

$$\text{R} \cdot \frac{\frac{d^2y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} dx = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx.$$

$$\therefore \text{R} = \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

This gives the value for the radius of curvature R, at the point (x, y) on the curve, in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Exercise.—Determine the radius of curvature at the point x, y, on the parabola $y^2 = 4ax$.

$$y = 2\sqrt{a}\cdot x^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{x^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2} = -\tfrac{1}{2} \cdot \frac{\sqrt{a}}{(x)^{\frac{3}{2}}}.$$

$$\therefore \text{R} = \frac{\left\{ 1 + \frac{a}{x} \right\}^{\frac{3}{2}}}{-\frac{1}{2}\frac{\sqrt{a}}{x^{\frac{3}{2}}}} = -\frac{\frac{(x+a)^{\frac{3}{2}}}{x^{\frac{3}{2}}}}{\frac{\sqrt{a}}{2} \cdot \frac{1}{x^{\frac{3}{2}}}} = -\frac{2(x+a)^{\frac{3}{2}}}{\sqrt{a}},$$

The meaning of the negative sign in the above result should be noticed. In Fig. 23, the curve ABE is convex toward the x axis, and θ is consequently increasing, or $d\theta$ is positive. Since a positive sign has been implicitly affixed to the quantity $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$ on the right-hand side of the equation, we must always have the product $\mathrm{R}d\theta$ positive. In other words, when θ increases with x (and $d\theta$ is consequently positive), R is positive; when θ decreases with x (and $d\theta$ is negative, the curve $y=f(x)$ being then concave toward the x axis), R must be negative.

It can easily be seen, by reference to Fig. 9, page 22, that the parabola

$$y^2 = 4ax$$

is concave toward the x axis, which accounts for the negative sign prefixed to the radius of curvature.

We can say, generally, that when the centre of curvature is on the same side of the curve as the axis of x, R is negative; otherwise R is positive.

Exercise.—Determine the radii of curvature of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the extremities of the major and minor axes.

$$y=b\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}},$$

$$\frac{dy}{dx}=-\frac{b}{a}\cdot\frac{x}{(a^2-x^2)^{\frac{1}{2}}},$$

$$\frac{d^2y}{dx^2}=-\frac{b}{a}\cdot\frac{(a^2-x^2)^{\frac{1}{2}}+\dfrac{x^2}{(a^2-x^2)^{\frac{1}{2}}}}{(a^2-x^2)} \text{ (compare p. 49)}$$

$$=-\frac{b}{a}\frac{a^2-x^2+x^2}{(a^2-x^2)^{\frac{3}{2}}}=\frac{-ab}{(a^2-x^2)^{\frac{3}{2}}}.$$

$$\mathrm{R}=\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}=\frac{\left\{1+\dfrac{b^2}{a^2}\cdot\dfrac{x^2}{(a^2-x^2)}\right\}^{\frac{3}{2}}}{\dfrac{-ab}{(a^2-x^2)^{\frac{3}{2}}}}=\frac{\dfrac{\{a^4+(b^2-a^2)x^2\}^{\frac{3}{2}}}{a^3(a^2-x^2)^{\frac{3}{2}}}}{\dfrac{-ab}{(a^2-x^2)^{\frac{3}{2}}}}$$

$$=-\frac{\{a^4+(b^2-a^2)x^2\}^{\frac{3}{2}}}{a^4b}.$$

If b is the minor semi-axis, and a the major semi-axis, then the abscissa of the end of the minor semi-axis is $x=0$. At this point,

$$\mathrm{R}=-\frac{a^6}{a^4b}=-\frac{a^2}{b}.$$

The abscissa of the end of the major axis is $x=a$. At this point,

$$\mathrm{R}=-\frac{\{a^4+(b^2-a^2)a^2\}^{\frac{3}{2}}}{a^4b}=-\frac{a^3b^3}{a^4b}=-\frac{b^2}{a}.$$

Remark.—It should be noticed that in the case of a curve which is only very slightly inclined to the axis of x, $\frac{dy}{dx}$ will be approximately equal to zero. In this case the radius of curvature is given by the approximate equation

$$\mathrm{R}=\frac{1}{\frac{d^2y}{dx^2}}.$$

This result is of some importance, and is frequently used in connection with problems in physics and mechanics.

Exercises.—(1.) Determine the radius of curvature to the curve $y=\sqrt{\frac{5}{3}}\cdot x^3$ at the point corresponding to the abscissa $x=1$.

Answer, $\mathrm{R}=\frac{32}{\sqrt{15}}$.

(2.) Determine the radius of curvature at any point of the curve

$$y=\frac{c}{2}\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right).$$

Answer, $\mathrm{R}=\frac{c}{4}\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right)^2$.

(3.) The strength (not stiffness) of a rectangular beam varies directly as the product of its breadth and the square of its depth. What must be the breadth of the strongest rectangular beam which can be cut from a circular trunk of radius a?

(Compare page 65.) *Answer*, $\frac{a}{\sqrt{3}}$.

CHAPTER VI.

INTEGRATION.

LET AB (Fig. 24) represent part of a curve, of which the analytical expression is $y=f(x)$. Draw a number of ordinates, CD, EF, GH . . . . RS. Then if these ordinates are drawn sufficiently close to each other, DF, FH . . . . will represent successive increments of x, and each may therefore be represented by the symbol dx.

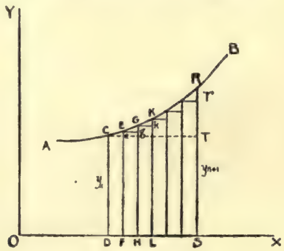

FIG. 24.

Draw Ce, Eg, Gk . . . . parallel to the axis of x. Then, as already explained, eE, gG, kK . . . . will represent successive increments of y, and each may be represented by the symbol dy. If we wish to specify any particular increment of y, such, for instance, as eE, we must use some convention to distinguish that particular increment of y from others. We may, for instance, write $e\mathrm{E}=dy_1$, $g\mathrm{G}=dy_2$ . . . ., DF being at the same time written dx_1, and $\mathrm{FH}=dx_2$ . . . .

Suppose we were required to find the sum of the increments of y, commencing with eE, and proceeding along the curve, till the increment rR is reached. We might write the result—

$$e\mathrm{E}+g\mathrm{G}+k\mathrm{K}+\ \ldots\ldots\ +r\mathrm{R},$$

or as

$$dy_1+dy_2+dy_3+\ \ldots\ldots\ +dy_n,$$

if $r\mathsf{R}=dy_n$. A simpler method, however, would be to write the result—

Sum of successive values of dy between the limits $y=y_1$ and $y=y_{n+1}$,

where $y_1=\mathsf{DC}$, and $y_{n+1}=\mathsf{RS}$.

The conventional method of writing this is—

$$\int_{y_1}^{y_{n+1}} dy$$

where $\int$ stands for "the sum of successive values of," and y_1 and y_{n+1} represent the values of the limiting ordinates $\mathsf{DC}=y_1$, $\mathsf{SR}=y_{n+1}$.

Through C draw the straight line CT parallel to OX, and cutting the ordinate SR in T.

Then $\mathsf{TR}=\mathsf{SR}-\mathsf{DC}=y_{n+1}-y_1$. Also, TR is obviously equal to the sum of the increments of y—

$$e\mathsf{E}+g\mathsf{G}+k\mathsf{K}+\ \ldots\ +r\mathsf{R}.$$

Hence we may write—

$$\int_{y_1}^{y_{n+1}} dy=y_{n+1}-y_1 \qquad . \qquad . \qquad . \qquad (1).$$

But we have assumed that $y=f(x)$. If $\mathsf{C}e=\mathsf{DF}=dx$ is infinitely small, then $\mathsf{E}e\mathsf{C}$ will be approximately a right-angled triangle, and $\frac{e\mathsf{E}}{\mathsf{C}e}=\tan\theta$, if $\angle \mathsf{EC}e=\theta$.

$$\therefore\ e\mathsf{E}=dy=\mathsf{C}e\tan\theta=dx\tan\theta.$$

But, as proved on page 28, $\tan\theta=\frac{dy}{dx}=f'(x)$.

$$\therefore\ dy=\frac{dy}{dx}\cdot dx=f'(x)dx.$$

Let $\mathsf{OD}=x_1$, $\mathsf{OS}=x_{n+1}$. Then y_1 will be equal to the value of $f(x)$, when x_1 is substituted for x. Similarly, y_{n+1} will be equal to the value of $f(x)$, when $x=x_{n+1}$.

$\therefore\ y_{n+1}-y_n=$ (the value of $f(x)$, when $x=x_{n+1}$) − (the value of $f(x)$, when $x=x_1$).

The member to the right of the last equation is conventionally written—

$$\left[f(x) \right]_{x=x_1}^{x=x_{n+1}}.$$

(1) can now be rewritten—

$$\int_{x=x_1}^{x=x_{n+1}} f'(x)dx = \left[f(x) \right]_{x=x_1}^{x=x_{n+1}} \quad . \quad . \quad (2).$$

The meaning of (2) must now be carefully considered. The symbol $\int$ is termed the *sign of integration*, and the quantity to the right of (2) is said to be the *definite integral* of $f'(x)dx$, the quantity under the sign of integration.

But $f'(x)$ is the first differential coefficient of $f(x)$. Therefore, in passing from $f'(x)$ (which is supposed to be given) to $f(x)$ (which is required), we must perform an operation exactly the reverse of differentiation. We then substitute the upper limiting value of x (namely, x_{n+1}) in the value of $f(x)$ so found, and subtract from this the value of $f(x)$ when the lower limiting value of x (namely, x_1) is substituted.

On page 41 a list of the simpler functions, together with their respective differential coefficients, is given. If, in a given expression placed under the sign of integration, we have dx multiplied by a function agreeing with any of the differential coefficients occurring on page 41, then we may at once write down $f(x)$ within the square brackets to the right of (2), as the function which, when differentiated, gives the said differential coefficient.

Take, for instance, the expression

$$\int_b^a \frac{1}{x}\, dx.$$

The quantity by which dx is multiplied is $\frac{1}{x}$, and on referring to page 41, it is at once seen that $\frac{1}{x}$ is the differential coefficient of $\log_\epsilon x$.

$$\therefore \int_b^a \frac{1}{x}\, dx = \left[\log_\epsilon x \right]_{x=b}^{x=a} = \log_\epsilon a - \log_\epsilon b = \log_\epsilon \frac{a}{b}.$$

We may also write—

$$\int \frac{1}{x}\,dx = \log_\epsilon x,$$

the result obtained being termed the *indefinite integral* of

$$\int \frac{1}{x}\,dx.$$

Thus the process of finding the indefinite integral of a function is equivalent to reversing the process of differentiation. The definite integral is obtained by substituting the limiting values of x, and proceeding as already described.

From the table of differential coefficients given on page 41, we may now draw up a table of indefinite integrals of the simpler functions.

Sometimes it is necessary to multiply and divide the quantity to be integrated by a constant, in order to bring the quantity under the sign of integration under one of the standard forms. Thus, if

$$y = x^{n+1},$$

$$\frac{dy}{dx} = (n+1)x^n.$$

$$\therefore \int x^n dx = \frac{1}{n+1}\int (n+1)x^n dx = \frac{1}{n+1}x^{n+1}.$$

If

$$y = \sin ax,$$

$$\frac{dy}{dx} = a \cos ax.$$

$$\therefore \int \cos ax\,dx = \frac{1}{a}\int a \cos ax\,dx = \frac{1}{a}\sin ax.$$

The student should now have no difficulty in verifying the following standard forms of integrals :—

$$\text{I.} \quad \int x^n dx \qquad = \frac{1}{n+1}\,x^{n+1},$$

where n may have any value, positive or negative, integral or fractional, except -1.

II. $$\int x^{-1}dx = \int \frac{dx}{x} = \log_\epsilon x.$$

III. $$\int a^{bx}dx = \frac{1}{b\log_\epsilon a}\cdot a^{bx}.$$

In particular,

$$\int \epsilon^{bx}dx = \frac{1}{b}\epsilon^{bx},$$

IV. $$\int \cos bx = \frac{1}{b}\sin bx.$$

V. $$\int \sin bx = -\frac{1}{b}\cos bx.$$

VI. $$\int \frac{dx}{\cos^2 bx} = \frac{1}{b}\tan bx.$$

VII. $$\int \frac{dx}{\sin^2 bx} = -\frac{1}{b}\cot bx.$$

VIII. $$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \text{ or } -\frac{1}{a}\cot^{-1}\frac{x}{a}.$$ (See p. 44.)

IX. $$\int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a}, \text{ or } -\cos^{-1}\frac{x}{a}.$$ (See page 45.)

X. $$\int \frac{dx}{\sqrt{(x^2+a^2)}} = \log_\epsilon\{x+\sqrt{x^2+a^2}\}.$$ (See page 52.)

XI. $$\int \frac{dx}{\sqrt{(x^2-a^2)}} = \log_\epsilon\{x+\sqrt{x^2-a^2}\}.$$ (See page 47.)

Exercises.

(1.) Find the numerical value of $\int_0^{\frac{\pi}{2}} \cos 3x\, dx$

$$\int_0^{\frac{\pi}{2}} \cos 3x = \frac{1}{3}\Big[\sin 3x\Big]_0^{\frac{\pi}{2}} = \frac{1}{3}\left\{\sin\frac{3\pi}{2} - \sin 0\right\}$$

$$= \frac{1}{3}\left\{-1-0\right\} = -\frac{1}{3}.$$

(2.) Find the numerical value of $\int_1^2 \frac{dx}{3x}$.

$$\int \frac{dx}{3x} = \frac{1}{3}\int\frac{dx}{x} = \frac{1}{3}\log_\epsilon x.$$

$$\therefore \int_1^2 \frac{dx}{3x} = \frac{1}{3}\Big[\log_\epsilon x\Big]_1^2 = \frac{1}{3}\left\{\log_\epsilon 2 - \log_\epsilon 1\right\}$$

$$= \frac{1}{3}\log_\epsilon\frac{2}{1} = \frac{1}{3}\log_\epsilon 2.$$

Let $\xi = \log_\epsilon 2$.

Then, by definition, $\epsilon^\xi = 2$.

Take the logarithms, to the base 10, of both sides of this equation.

$$\xi \log_{10}\epsilon = \log_{10} 2.$$

$$\therefore \xi = \frac{\log_{10} 2}{\log_{10}\epsilon},$$

$$\epsilon = 2{\cdot}71828,$$

$$\log_{10} 2{\cdot}71828 = 0{\cdot}43429,$$

$$\frac{1}{{\cdot}43429} = 2{\cdot}30258.$$

Hence,

$$\xi = \log_\epsilon 2 = \frac{1}{\log_{10}\epsilon} \times \log_{10} 2$$

$$= 2{\cdot}303 \times {\cdot}3010$$

$$= {\cdot}6932.$$

$$\therefore \int_1^2 \frac{dx}{3x} = \frac{\cdot 6932}{3} = \cdot 2310.$$

(3.) Find the value of $\int_{\frac{1}{2}}^{1} \epsilon^{2x} dx$.

$$\int \epsilon^{2x} dx = \frac{1}{2}\int 2\epsilon^{2x} dx = \frac{1}{2}\epsilon^{2x},$$

$$\int_{\frac{1}{2}}^{1} \epsilon^{2x} dx = \left[\frac{1}{2}\epsilon^{2x}\right]_{\frac{1}{2}}^{1} = \frac{1}{2}\left\{\epsilon^2 - \epsilon\right\}$$

$$= \frac{1}{2} \times 2\cdot 718\left\{2\cdot 718 - 1\right\}$$

$$= 2\cdot 335.$$

(4.) Find the value of $\int_1^3 5^{3x} dx$.

$$\int_{\frac{1}{3}}^{5} 5^{3x} dx = \frac{1}{3 \times \log_\epsilon 5} \cdot \left[5^{3x}\right]_{\frac{1}{3}}^{5} = \frac{1}{3 \log_\epsilon 5}\left\{5^{15} - 5\right\},$$

$\log_\epsilon 5 = 2\cdot 302 \times \log_{10} 5$ (see Exercise 2 above)

$= 2\cdot 302 \times \cdot 6990 = 1\cdot 609.$

$$\begin{array}{rr}
\log_{10} 5 = & \cdot 6990 \\
 & 15 \\
\hline
 & 34950 \\
 & 6990 \\
\hline
\log_{10} 5^{15} = & 10\cdot 4850
\end{array}
\qquad \therefore\ 5^{15} = 3\cdot 055 \times 10^{10}.$$

We may neglect 5 in comparison with $3\cdot 055 \times 10^{10}$.

$\therefore\ 5^{15} - 5 = 3\cdot 055 \times 10^{10}$ approximately.

$$\therefore \int_{\frac{1}{3}}^{5} 5^{3x} dx = \frac{3\cdot 055 \times 10^{10}}{3 \times 1\cdot 609} = 6\cdot 33 \times 10^{9}.$$

(5.) Find the value of $\int_1^3 (\cdot 469)^x dx$.

$$\int_1^3 (\cdot 469)^x dx = \frac{1}{\log_e \cdot 469} \Big[(\cdot 469)^x \Big]_1^3$$

$$= \frac{1}{\log_e \cdot 469} \cdot \left\{ (\cdot 469)^3 - \cdot 469 \right\},$$

$$\log_e \cdot 469 = 2\cdot 302 \times \log_{10} \cdot 469,$$

$$\log_{10} \cdot 469 = \bar{1}\cdot 6712 = -1 + \cdot 6712$$

$$= -\cdot 3288.$$

$$\therefore \log_e \cdot 469 = 2\cdot 302 \times (-\cdot 3288) = -\cdot 757.$$

(The student should carefully follow the above calculation, noticing particularly that a common logarithm with a negative characteristic cannot be used in such cases.)

$$(\cdot 469)^3 - \cdot 469 = \cdot 103 - \cdot 469$$

$$= -\cdot 366.$$

$$\therefore \int_1^3 (\cdot 469)^x dx = \frac{-\cdot 366}{-\cdot 757} = \cdot 483.$$

Integrating Areas.—Let AB (Fig. 25) be part of a curve,

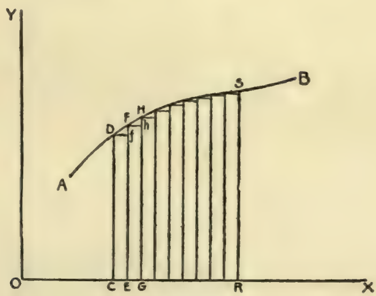

Fig. 25.

and let it be required to determine the area bounded by that curve, the two ordinates CD and RS, and the axis OX.

First divide the area into an indefinitely large number of elements, by drawing the ordinates EF, GH . . . . very close to each other. Then the area of the element, DFEC, is equal to $CD \times CE$, plus the area of the small, approximately triangular portion DFf. The area DFf will be very small in comparison with the area $DfEC$; and if CE is made indefinitely small, the area of DFf may be neglected.

Let $OC = x_1$, $CD = y_1$, $CE = dx_1$.

Then approximate area of $CDFE = y_1 dx_1$.

The area of each of the other rectangular strips can be obtained in a similar manner. To obtain the area of DSRC, we must obtain the sum of the various elements, any one of which is represented by the general expression ydx. Let $OR = x_2$. Then, from what has previously been said, it is clear that the required area is equal to

$$\int_{x_1}^{x_2} ydx,$$

x_1 and x_2 being the limiting values of x—namely, OC and OR.

If the equation of the curve AB is given in the form $y = f(x)$, we may write down the area of DSRC as

$$\int_{x_1}^{x_2} f(x)dx;$$

and if the quantity under the sign of integration can be integrated, the area can be found as required.

Example 1.—Let it be required to determine the area bounded by a straight line, AB (Fig. 26), represented by the equation $y = a + bx$ (see page 12), the two ordinates CA and DB, and the axis OX.

Let $OC = x_1$, $OD = x_2$.

$$\therefore \text{ Area } ABDC = \int_{x_1}^{x_2} ydx = \int_{x_1}^{x_2} (a + bx)dx.$$

When differentiating a quantity comprising a number of distinct functions added together, we found that we might differentiate each singly, and take the sum of the differential coefficients so obtained. In the same manner, when we have to integrate a quantity comprising a number of distinct functions added together, we may integrate each separately, and then obtain the sum of the resulting integrals. This result follows from the fact that an integration is the reverse of a differentiation. Thus—

$$\int(a+bx)dx=\int adx+\int bxdx.$$

$$\int adx=a\int dx=ax.\quad \text{(See the beginning of this chapter.)}$$

$$\int bxdx=\frac{b}{2}\int 2xdx=\frac{b}{2}x^2.$$

$$\therefore\ \int_{x_1}^{x_2}(a+bx)dx={}_{x_1}^{x_2}\left[ax+\frac{b}{2}x^2\right]=a(x_2-x_1)+\frac{b}{2}(x_2{}^2-x_1{}^2).$$

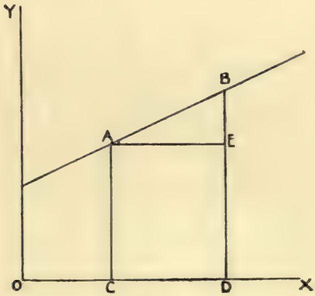

FIG. 26.

This result can also be written—

$$(x_2-x_1)\left\{a+\frac{b(x_2+x_1)}{2}\right\}.$$

The quantity within the first pair of brackets is equal to OD − OC = CD. The quantity within the second pair of brackets is equal to

$$\frac{1}{2}\left\{(a+bx_1)+(a+bx_2)\right\}=\frac{1}{2}(y_1+y_2).$$

This is equal to the mean of all the ordinates between the limits $x=x_1$ and $x=x_2$—that is, to the average height of the line AB above OX. Therefore,

Area ABDC = CD × average height of AB.

The student is, of course, familiar with this result. It is given here as a check on the accuracy of the method employed.

Example 2.—It is required to determine the area bounded by the curve $y=a \sin bx$ and the axis of x, between the limits $x=0$ and $x=\frac{\pi}{b}$.

Let the curve OBC (Fig. 27) represent the curve $y=a \sin bx$, between the limits $x=0$ and $x=\frac{\pi}{b}$.

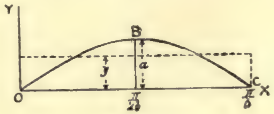

FIG. 27.

It is required to find the area bounded by OBC and the axis OC.

The maximum ordinate corresponds to a value of x given by the equation

$$bx=\frac{\pi}{2}. \qquad \therefore\ x=\frac{\pi}{2b}.$$

For this value of x, $\sin bx$ has its maximum value—namely, 1—and $y=a$.

The required area

$$=\int_0^{\frac{\pi}{b}} a \sin bx \cdot dx=\frac{a}{b}\int_0^{\frac{\pi}{b}} b \sin bx\, dx$$

$$=\frac{a}{b}\Big[-\cos bx\Big]_0^{\frac{\pi}{b}}=\frac{a}{b}\Big[1+1\Big]=\frac{2a}{b},$$

since $-\cos\pi=+1$, $+\cos 0=1$. (Compare with Exercise 3, p. 111.)

Example 3.—Find the value of the mean ordinate to the curve $y=a \sin bx$.

Let $\bar{y}$ be the mean value of the ordinates to the given curve, between the limits $x=0$ and $x=\frac{\pi}{b}$.

Then the area of the rectangle, bounded by (1) a straight line drawn parallel to OX, and at a distance $\bar{y}$ from it, (2) the axis OX, and (3) a line parallel to OY, drawn through C, and (4) the axis OY, must be equal to the area bounded by the curve OBC and the axis OC.

Area bounded by the above rectangle $= \bar{y} \times \text{OC} = \bar{y} \times \frac{\pi}{b}$.

Area bounded by curve OBC and axis $\text{OC} = \frac{2a}{b}$. (See previous example.) Hence,

$$\bar{y} \times \frac{\pi}{b} = \frac{2a}{b}.$$

$$\therefore \bar{y} = \frac{2a}{\pi} = \frac{a}{1\cdot5708}.$$

This result is of some importance in connection with alternating electrical currents. These fluctuate between a maximum value in one direction and an equal value in the opposite direction, and approximately follow a sine curve. In measuring the strengths of such currents, most instruments indicate the average value during half a complete period—that is, during the time in which the current continues flowing in one direction. If the maximum current is required, we can find it by multiplying the observed average current by $\frac{\pi}{2}$, or $1\cdot5708$; in other words, the maximum current is a little greater than one and a half times the average current.

Example 4.—Find the area bounded by a portion of the parabola $y^2 = 4ax$, the ordinates corresponding to the abscissæ x_1 and x_2, and the axis of x.

$$\text{The required area} = \int_{x_1}^{x_2} y\,dx = \int_{x_1}^{x_2} 2a^{\frac{1}{2}}x^{\frac{1}{2}}dx$$

$$= \frac{2}{3} \times 2a^{\frac{1}{2}} \int_{x_1}^{x_2} \frac{3}{2}x^{\frac{1}{2}}dx = \left[\frac{4a^{\frac{1}{2}}}{3} x^{\frac{3}{2}}\right]_{x_1}^{x_2},$$

since we increase the coefficient of x by 1 when integrating. (See page 93.)

$$\therefore \text{Required area} = \frac{4a^{\frac{1}{2}}}{3} \cdot \left\{x_2^{\frac{3}{2}} - x_1^{\frac{3}{2}}\right\}.$$

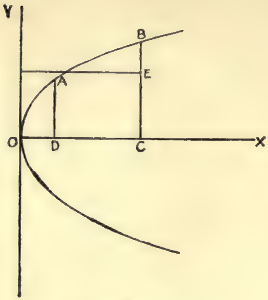

FIG. 28.

If $x_1 = 0$ (that is, if we find the area of OBC, Fig. 28), the result reduces to

$$\frac{4a^{\frac{1}{2}}}{3}x_2^{\frac{3}{2}} = \frac{2}{3} \times 2a^{\frac{1}{2}}x_2^{\frac{1}{2}} \times x_2 = \frac{2}{3} \times y_2 \times x_2$$
$$= \mathrm{OC} \times \mathrm{CE} \text{ (Fig. 28)},$$

where $$\mathrm{OC} = x_2,\ \mathrm{CE} = \frac{2}{3}\mathrm{CB} = \frac{2}{3}y_2.$$

Example 5.—Find the area of a circle of radius r.

Let ABCD be a circle, of which O is the centre, the radius OA being equal to r. Draw any two radii, OB, OC, very close together, and let it be agreed that angles shall be measured from the datum line OA in a sense opposite to that in which the hands of a clock revolve. Then we may write $\angle \mathrm{AOB} = \theta$, $\angle \mathrm{BOC} = d\theta$.

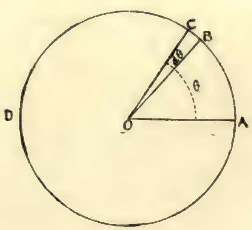

FIG. 29.

As $d\theta$ is indefinitely diminished, BOC will approximate more and more closely to an isosceles triangle, the length of the base of which is equal to $rd\theta$, and the altitude of which is equal to r. Then, from the well-known relation,

$$\text{Area of a triangle} = \frac{1}{2} \text{ base} \times \text{altitude},$$

we have, $$\text{Area of OBC} = \frac{1}{2} rd\theta \times r = \frac{1}{2} r^2 d\theta.$$

Since we wish to determine the area of the complete circle, we must integrate the above quantity between the limits $\theta = 0$ and $\theta = 2\pi$; in other words, we must suppose the radius OB to sweep out the entire area of the circle.

$\therefore$ Area of circle ABCD

$$= \int_0^{2\pi} \frac{r^2}{2} d\theta = \frac{r^2}{2} \int_0^{2\pi} d\theta = \frac{r^2}{2} \Big[\,\theta\,\Big]_0^{2\pi} = \frac{r^2}{2}\Big(2\pi - 0\Big) = \pi r^2.$$

In the above operation r^2 is removed from under the sign of integration, since its value is constant.

Example 6.—To find the length of the circumference of a circle.

By the definition of circular measure, length of an arc = radius of circle × circular measure of angle subtended by arc at centre of circle.

$\therefore$ In Fig. 29, length $\text{BC} = rd\theta$.

$\therefore$ Length of circumference of circle

$$= \int_0^{2\pi} rd\theta = r \Big[\,\theta\,\Big]_0^{2\pi} = 2\pi r.$$

Example 7.—To find the area of the surface of a sphere.

Let Fig. 30 represent a perspective view of a sphere, of which ABC is an equatorial section, D and E being poles, and FGH, KLM sections of the sphere by two planes, very close together, and parallel to the equatorial plane ABC. Join the centre O of the sphere to a point M on the section KLM, and let the angle $\text{MOD} = \theta$. If a point H on the section FGH, and in the same plane as MOD, be joined to the centre of the sphere, then we may obviously represent the angle MOH by $d\theta$. The breadth MH of the strip of surface between the sections FGH and KLM is obviously equal to $rd\theta$.

$$\text{The length of the strip} = 2\pi \times \text{PM}$$
$$= 2\pi \times r \sin\,\theta$$

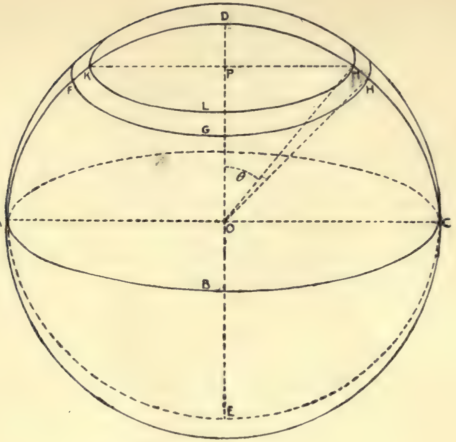

FIG. 30.

to a first approximation. (Since the strip is narrow, the lengths of its two edges may be considered to be equal.)

$$\therefore \text{ Area of strip} = 2\pi r \sin \theta \times r d\theta$$
$$= 2\pi r^2 \sin \theta d\theta.$$

If we integrate this value from $\theta = 0$ to $\theta = \frac{\pi}{2}$, we shall sum up the areas of all such strips comprised in the upper half of the spherical surface.

This result multiplied by 2 will give the area of the complete surface of the sphere.

$\therefore$ Area of surface of sphere

$$= 2 \times \int_0^{\frac{\pi}{2}} 2\pi r^2 \sin \theta d\theta$$

$$= 4\pi r^2 \int_0^{\frac{\pi}{2}} \sin \theta d\theta = 4\pi r^2 \Big[-\cos \theta \Big]_0^{\frac{\pi}{2}}$$

$$= 4\pi r^2 \{ -0 - (-1) \} = 4\pi r^2.$$

Example 8.—To find the volume of a solid sphere.

We may here consider the solid sphere to be cut up into an indefinitely large number of slices by planes similar to those of which the sections are FGH and KLM (Fig. 30). If the planes are indefinitely close to each other, the areas of opposite faces of one of these slices will be approximately equal. Then the volume of a slice = area of one of its circular faces × thickness of slice.

Referring to Fig. 30, and using the same notation as in the previous example, we have—

$$\text{Area of section KLM} = \pi \times (\text{PM})^2 = \pi \times r^2\sin^2\theta.$$
$$\text{Thickness of a slice} \quad = r d\theta \times \sin\,\theta.$$

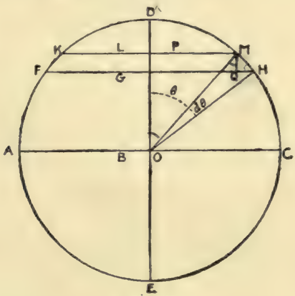

Fig. 31.

This last result will be obvious on reference to Fig. 31. The lettering corresponds with Fig. 30. Treating the arc MH as a straight line, we see that in the right-angled triangle MHQ, $\angle \text{MHQ} = \frac{\pi}{2} - \angle \text{HMQ} = \angle \text{OMQ}$, since OM is perpendicular to MH. Also, since MQ, OD are parallel, and MO intersects them, $\angle \text{OMQ} = \angle\ \text{MOD} = \theta$. Hence $\angle\ \text{MHQ} = \theta$. The thickness MQ of the slice $= \text{MH} \sin\ \text{MHQ} = r d\theta \times \sin\ \theta$.

$$\therefore \text{ Volume of slice} = \pi r^2 \sin^2\theta \times r \sin\,\theta d\theta$$
$$= \pi r^3 \sin^3\theta d\theta.$$

Integrating this between the limits $\theta = 0$ and $\theta = \frac{\pi}{2}$, we find the volume of half of the sphere. Hence volume of sphere

$$= 2 \times \int_0^{\frac{\pi}{2}} \pi r^3 \sin^3\theta d\theta$$

$$= 2\pi r^3 \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta.$$

A simple transformation must be effected in order to bring the expression under the sign of integration into a form which can readily be dealt with.

$$\sin^3\theta = \sin^2\theta \, . \sin\theta = (1 - \cos^2\theta) \sin\theta.$$

$$\therefore \int \sin^3\theta d\theta = \int (1 - \cos^2\theta) \sin\theta d\theta = \int \sin\theta d\theta - \int \cos^2\theta \sin\theta d\theta.$$

Now, $\int \sin\theta d\theta = -\cos\theta$. (See page 94.)

Also, if

$$y = \cos^3\theta,$$

$$\frac{dy}{d\theta} = -3\cos^2\theta \sin\theta d\theta.$$

$$\therefore \int \cos^2\theta \sin\theta d\theta = \frac{1}{3}\int 3\cos^2\theta \sin\theta d\theta$$

$$= -\frac{1}{3}\cos^3\theta.$$

$$\therefore \int \sin\theta d\theta - \int \cos^2\theta \sin\theta d\theta = -\cos\theta + \frac{1}{3}\cos^3\theta.$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta = {}_0^{\frac{\pi}{2}}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]$$

$$= \left\{-0 + \frac{0^3}{3} - (-1) - \frac{1}{3}(1)^3\right\}$$

$$= \frac{2}{3}.$$

$$\therefore \text{Volume of sphere} = 2\pi r^3 \times \frac{2}{3} = \frac{4}{3}\pi r^3.$$

Problem.—To determine the volume bounded by a paraboloid of revolution, and a plane drawn perpendicularly to the axis of the same.

Let AOB (Fig. 32) be part of the parabola represented by the equation

$$y^2 = 4ax.$$

The line OX is termed the axis of the parabola. If we suppose the curve AOB to revolve about the axis OX, a surface will be traced out in space which is termed a paraboloid of revolution.

Let ACB be the section of a plane perpendicular to the axis, and passing through C, a point on the latter, at a distance x_1 from the origin. It is required to determine the space bounded by the paraboloid of revolution and the plane of which ACB is a section.

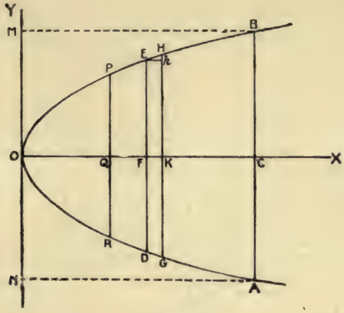

FIG. 32.

Imagine two planes, of which the sections are DFE and GKH, to be drawn very close to each other, and perpendicular to the axis OX; and let $\mathsf{OF}=x$, $\mathsf{FK}=dx$, $\mathsf{FE}=y$. It is obvious that these two planes will cut off a circular slice of the paraboloid of revolution, the thickness of the slice being $\mathsf{FK}=dx$, and the area of either face being approximately equal to $\pi\times(\mathsf{FE})^2=\pi y^2$. Hence the volume of the slice $=\pi y^2 dx$.

But $y^2=4ax$. $\therefore$ Volume of slice $=\pi\times 4axdx$.

If we integrate this quantity between the limits $x=0$ and $x=\mathsf{OC}=x_1$, we shall obtain the volume of the space bounded by the paraboloid of revolution, and the plane of which ACB is a section. Hence the required volume

$$=4a\pi\int_0^{x_1} xdx = 4a\pi \left[\frac{x^2}{2}\right]_0^{x_1} = 2a\pi x_1{}^2 = 2a\pi\times(\mathsf{OC})^2$$

$$=\frac{\pi(\mathsf{OC})}{2}\times 4a(\mathsf{OC})$$

$$=\frac{\pi\mathsf{OC}\cdot(\mathsf{BC})^2}{2},$$

since $(\mathsf{BC})^2=4a\mathsf{OC}$.

Also, $\pi(\mathsf{BC})^2\times\mathsf{OC}$ is equal to the volume bounded by a circular cylinder, formed by the revolution of the lines BM, AN about the axis OX, the sections of its circular ends coinciding with the lines

ACB and NOM. Hence the volumes bounded by the paraboloid of revolution of which the section is AOB, and the plane of which the section is ACB, is equal to half the volume of the circular cylinder, of which BANM is the longitudinal section.

Problem.—To determine the area of the curved surface of a right circular cone.

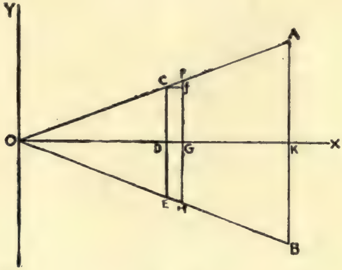

Fig. 33.

Let AOB (Fig. 33) be a section, through the axis OX, of the right circular cone. Then AK represents the radius of the circular base, and OK represents the altitude of the cone. Let $OK = x_1$, $\frac{AK}{OK} = a$.

Let CDE, FGH be sections of two planes, very close to each other, drawn parallel to the base of the cone; and let $OD = x$, $DG = dx$, $DC = y$.

Then, since the triangles CDO, AKO are similar, we have

$$\frac{y}{x} = \frac{AK}{OK} = a. \qquad \therefore\ y = ax.$$

The two planes of which the sections are CDE and FGH will cut off a narrow circular strip, of which the sections are CF and EH, from the curved surface of the cone. Since the planes are very close to each other, the length of either side of this strip may be taken as approximately equal to $2\pi \times DC = 2\pi y$. Hence the area of this strip $= 2\pi y \times (CF) = 2\pi ax \times (CF)$.

Draw Cf parallel to OX. Then $Cf = dx$; and since the triangles FfC, AKO are similar, $Ff = adx$.

$$\therefore\ CF = \{(dx)^2 + (adx)^2\}^{\frac{1}{2}} = (1 + a^2)^{\frac{1}{2}} dx.$$

Hence the area of the circular strip of which CF and EH are sections, is equal to $2\pi ax \times (1+a^2)^{\frac{1}{2}}dx$.

$\therefore$ Total area of curved surface

$$= 2\pi a(1+a^2)^{\frac{1}{2}}\int_0^{x_1} xdx = 2\pi a(1+a^2)^{\frac{1}{2}}\,{}_0^{x_1}\!\left[\frac{x^2}{2}\right] = \pi a(1+a^2)^{\frac{1}{2}}x_1^2.$$

But $$2\pi a x_1 = 2\pi a(\text{OK}) = 2\pi(\text{KA}),$$

$$(1+a^2)^{\frac{1}{2}}x_1 = (x_1^2 + a^2x_1^2)^{\frac{1}{2}} = (\text{OK}^2 + \text{KA}^2)^{\frac{1}{2}} = \text{OA}.$$

$\therefore$ Area of curved surface of cone $= \frac{1}{2} \times 2\pi(\text{KA}) \times \text{OA} =$ half the circumference of the base $\times$ the length of the sloping side of the cone.

Problem.—To determine the volume of a right circular cone.

The planes, of which the sections are EDC and HGF (Fig. 33), cut a circular slice out of the cone. The thickness of this slice is equal to dx; and since the planes are supposed to be very close to each other, the area of either of the plane faces may be taken as equal to $\pi(\text{DC}^2) = \pi y^2$.

$\therefore$ Volume of slice cut off $= \pi y^2 dx = \pi(ax)^2 dx = \pi a^2 x^2 dx$.

Let the altitude of the cone $= \text{OK} = x_1$.

Then total volume of cone

$$= \pi a^2 \int_0^{x_1} x^2 dx = \pi a^2\,{}_0^{x_1}\!\left[\frac{x^3}{3}\right] = \frac{\pi a^2 x_1^3}{3} = \pi a^2 x_1^2 \cdot \frac{x_1}{3} = \pi(\text{KA})^2 \times \frac{\text{OK}}{3}.$$

Hence the volume of a cone is equal to the area of its base multiplied by one-third of its altitude.

Problem.—To determine the area of the surface cut off from a paraboloid of revolution by two planes drawn perpendicular to the axis.

Let AOB (Fig. 32) be a section of the paraboloid of revolution, the equation to this section being $y^2 = 4ax$.

Let the sections of the two planes, drawn perpendicular to the axis OX, be the lines PQR and BCA respectively. It is required to determine the area of that part of the surface of which the sections are PEB and RDA.

Let EFD, HKG represent the sections of two planes drawn very close to each other, and let $\text{OF} = x$, $\text{FK} = dx$, $\text{FE} = y$. These two planes will cut a narrow circular strip off from the surface, the length of the strip being equal to $2\pi \times \mathsf{EF} = 2\pi y$ (since, if dx is very small, HK will be approximately equal to EF), whilst the breadth of the strip is equal to EH.

Draw Eh parallel to OX. Then E$h = dx$, H$h = dy$.

$$\therefore\ \mathsf{EH} = \sqrt{(dx)^2 + (dy)^2} = \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} \cdot dx.$$

Also, since $$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}.$$

$$\therefore\ \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}} = \left(1 + \frac{a}{x}\right)^{\frac{1}{2}} = \left(\frac{x+a}{x}\right)^{\frac{1}{2}},$$

$$\mathsf{EH} = \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx.$$

$$\therefore\ \text{Area of strip} = 2\pi y \times (\mathsf{EH}) = 2\pi \times 2a^{\frac{1}{2}}x^{\frac{1}{2}} \times \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx$$

$$= 4\pi a^{\frac{1}{2}}(x+a)^{\frac{1}{2}} dx.$$

To determine the area of that part of the surface of the paraboloid of revolution corresponding to the sections PB and RA, we must integrate the result just obtained between the limits $x = x_1$ and $x = x_2$.

$\therefore$ Required area

$$= \quad 4\pi a^{\frac{1}{2}} \int_{x_1}^{x_2} (x+a)^{\frac{1}{2}} dx$$

$$= \frac{2}{3} \times 4\pi a^{\frac{1}{2}} \int_{x_1}^{x_2} \frac{3}{2}(x+a)^{\frac{1}{2}} dx = \frac{8}{3}\pi a^{\frac{1}{2}} \Big[(x+a)^{\frac{3}{2}} \Big]_{x_1}^{x_2}$$

$$= \frac{8}{3}\pi a^{\frac{1}{2}} \left\{ (x_2+a)^{\frac{3}{2}} - (x_1+a)^{\frac{3}{2}} \right\}.$$

[The above result follows from the fact that if $y=(x+a)^{\frac{3}{2}}$,

$$\frac{dy}{dx}=\frac{3}{2}(x+a)^{\frac{1}{2}}.]$$

The area of the surface of the above paraboloid of revolution, extending from the vertex O to the section of the plane BCA, is equal to

$$\frac{8}{3}\pi a^{\frac{1}{2}}(x_2+a)^{\frac{3}{2}}.$$

Exercises.—(1.) Find the area enclosed by the catenary (see page 209) of which the equation is $y=\frac{c}{2}\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right)$, the ordinates corresponding to $x=0$ and $x=x_1$, and the axis of x.

Answer. $\frac{c^2}{2}\left(\epsilon^{\frac{x_1}{c}}-\epsilon^{-\frac{x_1}{c}}\right)$.

(2.) The catenary referred to in Exercise 1 is caused to rotate about the axis of x, thus generating a surface of revolution. Find the volume enclosed by this surface and two planes drawn through $x=+a$ and $x=-a$ respectively.

Answer. $\frac{\pi c^2}{4}\left\{c\left(\epsilon^{\frac{2a}{c}}-\epsilon^{-\frac{2a}{c}}\right)+4a\right\}$.

(3.) Find the area enclosed by the curve $y=a\cos x$, the ordinates corresponding to $x=0$ and $x=\pi$, and the axis of x.

Suggestion.—Notice that between $x=0$ and $x=\frac{\pi}{2}$, y is positive, whilst between $x=\frac{\pi}{2}$ and $x=\pi$, y is negative. (See Fig. 4, p. 14.) Hence, as x increases, ydx changes its sign when $x=\frac{\pi}{2}$. Hence $\int_0^{\pi}\cos x dx=0$. (Compare with Example 2, page 100.) Consequently the required area $=2a\int_0^{\frac{\pi}{2}}\cos x dx=2a$.

CHAPTER VII.

SPECIAL METHODS USED IN INTEGRATING.

AS explained in the preceding chapter, in order to effect the integration of a given function, we must find another function of which the differential coefficient is equal to the given function. If the given function to be integrated occurs in the list of standard forms given on page 94, the integration may be effected immediately. On the other hand, it often happens that a function to be integrated differs considerably from any of the forms given on page 94. In that case some special methods of integration must be used. Some of the most important methods to be used in this connection will now be described.

SUBSTITUTION.—A function which does not agree with any of the standard forms given on page 94 can sometimes be transformed into one of the standard forms. The general principles of the method to be used in such a case can be best understood by reference to the following examples:—

Example 1.—Find the value of

$$\int \frac{dx}{(2ax - x^2)^{\frac{1}{2}}}.$$

Now, $\quad 2ax - x^2 = a^2 - (a^2 - 2ax + x^2) = a^2 - (a - x)^2.$

Hence the given integral can be written,

$$\int \frac{dx}{\{a^2 - (a - x)^2\}^{\frac{1}{2}}}.$$

Let $a - x = z$. Then

$$-dx = dz.$$

Substituting in the above integral, it becomes transformed into

$$\int \frac{-dz}{(a^2-z^2)^{\frac{1}{2}}} = \cos^{-1}\frac{z}{a} = \cos^{-1}\frac{a-x}{a}.$$

Example 2.—Find the value of

$$\int \frac{dx}{x(2ax-a^2)^{\frac{1}{2}}}.$$

$$2ax-a^2 = x^2-(x^2-2ax+a^2) = x^2-(x-a)^2.$$

$$\therefore\ (2ax-a^2)^{\frac{1}{2}} = (x^2-(x-a)^2)^{\frac{1}{2}} = x\left(1-\left(\frac{x-a}{x}\right)^2\right)^{\frac{1}{2}}.$$

$$\therefore \int \frac{dx}{x(2ax-a^2)^{\frac{1}{2}}} = \int \frac{dx}{x^2\left(1-\left(\frac{x-a}{x}\right)^2\right)^{\frac{1}{2}}}.$$

Let $\qquad \dfrac{x-a}{x} = z.$

Then $\qquad 1-\dfrac{a}{x} = z. \quad \therefore\ \dfrac{a}{x} = 1-z,$

$$\text{and } x = \frac{a}{1-z}.$$

Taking the differentials of both sides, we get

$$dx = \frac{a}{(1-z)^2}dz$$

$$= \frac{x^2}{a}\cdot dz.$$

$$\therefore\ \frac{dx}{x^2} = \frac{dz}{a}.$$

$$\therefore \int \frac{dx}{x^2\left(1-\left(\frac{x-a}{x}\right)^2\right)} = \frac{1}{a}\int \frac{dz}{(1-z^2)^{\frac{1}{2}}} = \frac{1}{a}\sin^{-1}\frac{z}{1} = \frac{1}{a}\sin^{-1}\frac{x-a}{x}.$$

Example 3.—Find the value of $\displaystyle\int \frac{dx}{(1-5x-x^2)^{\frac{1}{2}}}.$

$$1-5x-x^2 = 1+\left(\frac{5}{2}\right)^2 - \left(\left(\frac{5}{2}\right)^2+5x+x^2\right) = \frac{29}{4}-\left(\frac{5}{2}+x\right)^2.$$

Let $\quad \frac{5}{2}+x=z$. Then $dx=dz$.

$$\int \frac{dx}{(1-5x-x^2)^{\frac{1}{2}}}=\int \frac{dx}{\left(\frac{29}{4}-z^2\right)^{\frac{1}{2}}}=\sin^{-1}\frac{z}{\frac{\sqrt{29}}{2}}$$

$$=\sin^{-1}\frac{\frac{5}{2}+x}{\frac{\sqrt{29}}{2}}=\sin^{-1}\frac{5+2x}{\sqrt{29}}.$$

Example 4.—Find the value of $\displaystyle\int \frac{dx}{(a+bx-cx^2)^{\frac{1}{2}}}$.

$$(a+bx-cx^2)^{\frac{1}{2}}=c^{\frac{1}{2}}\left(\frac{a}{c}+\frac{b}{c}x-x^2\right)^{\frac{1}{2}}$$

$$=c^{\frac{1}{2}}\left\{\frac{a}{c}+\left(\frac{b}{2c}\right)^2-\left(\left(\frac{b}{2c}\right)^2-\frac{b}{c}x+x^2\right)\right\}^{\frac{1}{2}}$$

$$=c^{\frac{1}{2}}\left\{\frac{a}{c}+\left(\frac{b}{2c}\right)^2-\left(x-\frac{b}{2c}\right)^2\right\}^{\frac{1}{2}}.$$

Let $\quad \frac{a}{c}+\left(\frac{b}{2c}\right)^2=\mathsf{A}^2. \quad \therefore \mathsf{A}=\frac{\sqrt{4ac+b^2}}{2c}.$

Let $\quad x-\frac{b}{2c}=z. \quad \therefore dx=dz.$

$$\therefore \int \frac{dx}{(a+bx-cx^2)^{\frac{1}{2}}}=\frac{1}{c^{\frac{1}{2}}}\int \frac{dz}{(\mathsf{A}^2-z^2)^{\frac{1}{2}}}=\frac{1}{\sqrt{c}}\cdot \sin^{-1}\frac{z}{\mathsf{A}}$$

$$=\frac{1}{\sqrt{c}}\sin^{-1}\frac{\frac{2cx-b}{2c}}{\frac{\sqrt{4ac+b^2}}{2c}}=\frac{1}{\sqrt{c}}\sin^{-1}\frac{2cx-b}{\sqrt{4ac+b^2}}.$$

Example 5.—Find the value of $\displaystyle\int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}}$.

By a similar train of reasoning to that employed in the preceding example, it may be shown that

$$(a+bx+cx^2)^{\frac{1}{2}} = c^{\frac{1}{2}}\left\{\frac{a}{c}-\left(\frac{b}{2c}\right)^2+\left(\frac{b}{2c}+x\right)^2\right\}^{\frac{1}{2}}.$$

Let $\quad \dfrac{a}{c}-\left(\dfrac{b}{2c}\right)^2 = \mathsf{A}^2. \quad \therefore\ \mathsf{A} = \dfrac{\sqrt{(4ac-b^2)}}{2c}.$

Let $\quad \dfrac{b}{2c}+x = z. \quad \therefore\ dx = dz.$

Also, $\quad \mathsf{A}^2 + z^2 = \dfrac{1}{c}(a+bx+cx^2).$

$$\therefore \int \frac{dx}{(a+bx+cx^2)^{\frac{1}{2}}} = \frac{1}{\sqrt{c}}\int\frac{dz}{(\mathsf{A}^2+z^2)^{\frac{1}{2}}}$$

$$= \frac{1}{\sqrt{c}}\log_\epsilon\{z+\sqrt{z^2+\mathsf{A}^2}\} \quad \text{(see page 94)}$$

$$= \frac{1}{\sqrt{c}}\cdot\log_\epsilon\left\{\frac{2cx+b}{2c}+\frac{\sqrt{a+bx+cx^2}}{\sqrt{c}}\right\}$$

$$= \frac{1}{\sqrt{c}}\log_\epsilon\{2cx+b+2\sqrt{c}\sqrt{a+bx+cx^2}\} - \frac{1}{\sqrt{c}}\log_\epsilon 2c.$$

Example 6.—Find the value of $\displaystyle\int\frac{dx}{x^2-a^2}$.

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right).$$

$$\therefore \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\left\{\int\frac{dx}{x-a}-\int\frac{dx}{x+a}\right\}.$$

Let $\quad x-a=z. \quad$ Then $dx = dz.$

$$\int\frac{dx}{x-a} = \int\frac{dz}{z} = \log_\epsilon z = \log_\epsilon(x-a).$$

Similarly, $\quad \displaystyle\int\frac{dx}{x+a} = \log_\epsilon(x+a).$

$$\therefore \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\{\log_\epsilon(x-a)-\log_\epsilon(x+a)\}$$

$$= \frac{1}{2a}\log_\epsilon\frac{x-a}{x+a}.$$

The above solution is only possible when $\frac{x-a}{x+a}$ is positive, since it is impossible to obtain the logarithm of a negative quantity. If, on the other hand, $x<a$, we may write—

$$\int\frac{dx}{x^2-a^2}=-\int\frac{dx}{a^2-x^2}=-\frac{1}{2a}\left\{\int\frac{dx}{a-x}+\int\frac{dx}{a+x}\right\}$$

$$=-\frac{1}{2a}\{-\log_\epsilon(a-x)+\log_\epsilon(a+x)\}$$

$$=\frac{1}{2a}\log_\epsilon\frac{a-x}{a+x}.$$

Example 7.—Find the value of $\int\frac{dx}{a+bx+cx^2}$.

$$a+bx+cx^2=c\left\{\frac{a}{c}-\left(\frac{b}{2c}\right)^2+\left(\frac{b}{2c}+x\right)^2\right\}.$$

Two cases may now arise.

(a) If $4ac>b^2$, let $\frac{a}{c}-\left(\frac{b}{2c}\right)^2=\frac{4ac-b^2}{4c^2}=\mathsf{A}^2$. Also, let $\frac{b}{2c}+x=z$. $\therefore dx=dz$. Then

$$\int\frac{dx}{a+bx+cx^2}=\frac{1}{c}\int\frac{dz}{\mathsf{A}^2+z^2}=\frac{1}{\mathsf{A}c}\tan^{-1}\frac{z}{\mathsf{A}}$$

$$=\frac{2}{\sqrt{4ac-b^2}}\tan^{-1}\frac{2cx+b}{\sqrt{4ac-b^2}}.$$

(b) If $4ac<b^2$, let $\frac{a}{c}-\left(\frac{b}{2c}\right)^2=\frac{4ac-b^2}{4c^2}=-\mathsf{A}^2$. Then, with the same value of z as previously, we have—

$$\int\frac{dx}{a+bx+cx^2}=\frac{1}{c}\int\frac{dz}{z^2-\mathsf{A}^2}=\frac{1}{2\mathsf{A}c}\log\frac{z-\mathsf{A}}{z+\mathsf{A}}$$

$$=\frac{1}{\sqrt{(b^2-4ac)}}\log_\epsilon\frac{2cx+b-\sqrt{b^2-4ac}}{2cx+b+\sqrt{b^2+4ac}}.$$

(See Example 6, page 115.)

INTEGRATION BY PARTS.—If ξ and η are functions of x, then it has been shown (page 48) that

$$\frac{d(\xi\cdot\eta)}{dx}=\eta\frac{d\xi}{dx}+\xi\frac{d\eta}{dx}.$$

Multiply both sides of this equation by dx, and integrate. We thus get

$$\int \frac{d(\xi \cdot \eta)}{dx}\,dx = \int \eta\,\frac{d\xi}{dx}\,dx + \int \xi \cdot \frac{d\eta}{dx} \cdot dx.$$

The integral on the left-hand side of this equation is obviously equal to the product of ξ and η.

$$\therefore\ \xi\eta = \int \eta\,\frac{d\xi}{dx}\,dx + \int \xi\,\frac{d\eta}{dx}\,dx.$$

$$\therefore\ \int \xi\,\frac{d\eta}{dx}\,dx = \xi\eta - \int \eta\,\frac{d\xi}{dx}\,dx \quad . \quad . \quad . \quad (1).$$

Now it sometimes happens that a quantity under the sign of integration is formed from the product of two functions, one of which can be seen at a glance to be equal to the differential coefficient of a known function, whilst the other function is not directly integrable. Let the function which is equal to the differential coefficient of one of the standard forms be called $\frac{d\eta}{dx}$. Then η is known. Also, we can always differentiate the other function, which we call ξ. Then the quantity to be integrated may be represented by the member on the left-hand side of (1), and this may, if we please, be transformed into the expression represented by the right-hand side of (1). A little practice will enable the student to judge whether any advantage can be gained by this procedure. A number of problems in which this process, termed integration by parts, can advantageously be used, will now be solved.

Example 1.—Find the value of $\int x \log_\epsilon x dx$.

Here, under the sign of integration, we have the product of two functions, one of which (namely, x) can immediately be integrated, whilst the other (namely, $\log_\epsilon x$) is not equal to the differential coefficient of any simple function.

In (1) above, let $\quad \frac{d\eta}{dx} = x. \qquad \therefore\ \eta = \frac{x^2}{2}.$

Let $\quad \xi = \log_\epsilon x. \qquad \therefore\ \frac{d\xi}{dx} = \frac{1}{x}.$

Then, substituting in (1) above, we get

$$\int \log_\epsilon x \cdot x \cdot dx = \int \xi \frac{d\eta}{dx} dx = \xi\eta - \int \eta \frac{d\xi}{dx} dx$$
$$= \log_\epsilon x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} \cdot dx$$
$$= \frac{x^2}{2} \log_\epsilon x - \frac{1}{2} \int x dx$$
$$= \frac{x^2}{2} \log_\epsilon x - \frac{x^2}{4} = \frac{x^2}{2} \left(\log_\epsilon x - \frac{1}{2} \right).$$

Example 2.—Find the value of $\int x \sin ax\, dx$.

Here both functions, x and $\sin ax$, admit of immediate integration. On the other hand, if in (1) above we write $\frac{d\eta}{dx} = x$, $\xi = \sin ax$, we get

$$\int \xi \frac{d\eta}{dx} dx = \int \sin ax \cdot x \cdot dx = \eta\xi - \int \eta \frac{d\xi}{dx} dx$$
$$= \frac{x^2}{2} \sin ax - \int \frac{ax^2}{2} \cos ax \cdot dx.$$

The quantity under the sign of integration on the right of this equation is as far from an integrable form as ever.

Hence the transformation which we have effected is of no use to us. It may be noticed, however, that no matter how many times we may differentiate $\sin ax$, we shall never obtain a constant quantity as a result. On the other hand, if we differentiate x, we obtain 1 as a result. Hence, if we write $\xi = x$ in (1), x will no longer occur in the expression under the sign of integration to the right of (1), and this expression can consequently be immediately integrated.

Let $\qquad \frac{d\eta}{dx} = \sin ax. \quad \therefore \eta = -\frac{1}{a} \cos ax.$

Let $\qquad \xi = x. \qquad \therefore \frac{d\xi}{dx} = 1.$

$$\therefore \int \xi \frac{d\eta}{dx} dx = \int x \cdot \sin ax \cdot dx = \xi\eta - \int \eta \frac{d\xi}{dx} dx$$
$$= -\frac{x}{a} \cos ax - \int -\frac{\cos ax}{a} \times 1 \times dx$$
$$= -\frac{x}{a} \cos ax + \frac{\sin ax}{a^2}.$$

Example 3.—Find the value of $\int x^3 \sin ax\, dx$.

$$\int x^3 \sin ax\, dx = -\frac{x^3}{a}\cos ax - \int -\frac{3x^2}{a}\cdot\cos ax\, dx$$

$$= -\frac{x^3}{a}\cos ax + \frac{3}{a}\int x^2\cos ax\, dx.$$

Further, $\frac{3}{a}\int x^2\cos ax\, dx = \frac{3}{a^2}x^2\sin ax - \frac{2\cdot 3}{a^2}\int x\sin ax\, dx$

$$= \frac{3x^2}{a^2}\sin ax + \frac{2\cdot 3}{a^3}x\cos ax - \frac{2\cdot 3}{a^4}\sin ax.$$

$$\therefore \int x^3 \sin ax\, dx = -\frac{x^3}{a}\cos ax + \frac{3x^2}{a^2}\sin ax$$

$$+\frac{2\cdot 3\cdot x}{a^3}\cos ax - \frac{2\cdot 3}{a^4}\sin ax$$

$$= \left(-\frac{x^3}{a} + \frac{2\cdot 3\cdot x}{a^3}\right)\cos ax + \left(\frac{3x^2}{a^2} - \frac{2\cdot 3}{a^4}\right)\sin ax.$$

Example 4.—Find the value of $\int (a^2 - x^2)^{\frac{1}{2}}dx$.

The quantity under the sign of integration may be written $(a^2 - x^2)^{\frac{1}{2}} \times 1 \times dx$.

Let $\frac{d\eta}{dx} = 1. \quad \therefore \eta = x.$

Let $\xi = (a^2 - x^2)^{\frac{1}{2}}. \quad \therefore \frac{d\xi}{dx} = \frac{1}{2}\times(a^2 - x^2)^{-\frac{1}{2}}\times(-2x)$

$$= \frac{-x}{(a^2 - x^2)^{\frac{1}{2}}}.$$

$$\therefore \int (a^2 - x^2)^{\frac{1}{2}}dx = (a^2 - x^2)^{\frac{1}{2}}\cdot x + \int \frac{x^2 dx}{(a^2 - x^2)^{\frac{1}{2}}}.$$

But $\frac{x^2}{(a^2 - x^2)^{\frac{1}{2}}} = \frac{a^2 - (a^2 - x^2)}{(a^2 - x^2)^{\frac{1}{2}}} = \frac{a^2}{(a^2 - x^2)^{\frac{1}{2}}} - (a^2 - x^2)^{\frac{1}{2}}.$

$$\therefore \int (a^2 - x^2)^{\frac{1}{2}}dx = (a^2 - x^2)^{\frac{1}{2}}\cdot x + a^2\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} - \int (a^2 - x^2)^{\frac{1}{2}}dx.$$

Removing the integral to the extreme right of this equation to the left-hand side, we get

$$2\int (a^2-x^2)^{\frac{1}{2}}dx = (a^2-x^2)^{\frac{1}{2}}x + a^2\int \frac{dx}{(a^2-x^2)^{\frac{1}{2}}}.$$

The remaining integral to the right of this equation agrees with one of the standard forms. (See page 94.) Hence, finally, substituting the value of that integral, and dividing the equation through by 2, we get

$$\int (a^2-x^2)^{\frac{1}{2}}dx = \frac{x}{2}\cdot(a^2-x^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

Example 5.—Find the value of $\int \epsilon^{ax}\sin bx\,dx$.

Integrating by parts, we get

$$\int \epsilon^{ax}\sin bx\cdot dx = \frac{-\epsilon^{ax}}{b}\cos bx - \int \frac{-a\epsilon^{ax}}{b}\cos bx\,dx$$

$$= \frac{-\epsilon^{ax}}{b}\cos bx + \frac{a}{b}\int \epsilon^{ax}\cos bx\,dx.$$

Also, $\int \epsilon^{ax}\cos bx\cdot dx = \frac{\epsilon^{ax}\sin bx}{b} - \int \frac{a}{b}\epsilon^{ax}\sin bx\,dx.$

$$\therefore \int \epsilon^{ax}\sin bx\,dx = -\frac{\epsilon^{ax}}{b}\cos bx + \frac{a}{b^2}\epsilon^{ax}\sin bx - \frac{a^2}{b^2}\int \epsilon^{ax}\sin bx\,dx.$$

Removing the integral on the right of this latter equation to the left-hand side, and rearranging the remaining terms, we get

$$\left(1+\frac{a^2}{b^2}\right)\int \epsilon^{ax}\sin bx\,dx = \epsilon^{ax}\frac{a\sin bx - b\cos bx}{b^2}.$$

Dividing through by $1+\frac{a^2}{b^2} = \frac{a^2+b^2}{b^2}$, we get the required result—

$$\int \epsilon^{ax}\sin bx\,dx = \epsilon^{ax}\cdot\frac{a\sin bx - b\cos bx}{a^2+b^2}.$$

Example 6.—Prove, in a similar manner to that just used, that

$$\int \epsilon^{ax}\cos bx = \epsilon^{ax}\frac{a\cos bx + b\sin bx}{a^2+b^2}.$$

Example 7.—Find the value of $\int (x^2+a^2)^{\frac{1}{2}}dx$.

Integrating by parts, we get

$$\int (x^2+a^2)^{\frac{1}{2}}dx = (x^2+a^2)^{\frac{1}{2}}x - \int \frac{x^2dx}{(x^2+a^2)^{\frac{1}{2}}}.$$

But
$$\frac{x^2}{(x^2+a^2)^{\frac{1}{2}}} = \frac{x^2+a^2-a^2}{(x^2+a^2)^{\frac{1}{2}}} = (x^2+a^2)^{\frac{1}{2}} - \frac{a^2}{(x^2+a^2)^{\frac{1}{2}}}.$$

$$\therefore \int (x^2+a^2)^{\frac{1}{2}}dx = (x^2+a^2)^{\frac{1}{2}}x - \int (x^2+a^2)^{\frac{1}{2}}dx + a^2\int \frac{dx}{(x^2+a^2)^{\frac{1}{2}}}.$$

$$\therefore 2\int (x^2+a^2)^{\frac{1}{2}}dx = (x^2+a^2)^{\frac{1}{2}}x + a^2\int \frac{dx}{(x^2+a^2)^{\frac{1}{2}}}$$

$$= (x^2+a^2)^{\frac{1}{2}}x + a^2\log_\epsilon\{x+\sqrt{x^2+a^2}\}.$$

$$\therefore \int (x^2+a^2)^{\frac{1}{2}}dx = \frac{x}{2}(x^2+a^2)^{\frac{1}{2}} + \frac{a^2}{2}\log_\epsilon\{x+\sqrt{x^2+a^2}\}.$$

Example 8.—Find the value of $\int \left(\frac{a+x}{x}\right)^{\frac{1}{2}}dx$.

It must be our object, in this problem, to bring the quantity under the sign of integration to the form given in Example 7, by substituting a suitable value of x.

$$\int \left(\frac{a+x}{x}\right)^{\frac{1}{2}}dx = \int (a+x)^{\frac{1}{2}}\frac{dx}{\sqrt{x}}.$$

Let
$$x = z^2.$$

$$\therefore dx = 2z\,dz = 2\sqrt{x}\cdot dz. \quad \therefore \frac{dx}{\sqrt{x}} = 2dz.$$

$$\therefore \int (a+x)^{\frac{1}{2}}\frac{dx}{\sqrt{x}} = 2\int (a+z^2)^{\frac{1}{2}}dz = z(z^2+a)^{\frac{1}{2}} + a\log_\epsilon\{z+\sqrt{z^2+a}\}$$

$$= x^{\frac{1}{2}}(x+a)^{\frac{1}{2}} + a\log_\epsilon\{\sqrt{x}+\sqrt{x+a}\}$$

$$= \sqrt{(ax+x^2)} + a\log_\epsilon\{\sqrt{x}+\sqrt{x+a}\}.$$

Example 9.—Find the value of $\int \frac{dx}{\cos x}$.

$$\int \frac{dx}{\cos x} = \int \frac{\cos x \, dx}{\cos^2 x}.$$

Let $\qquad z = \sin x. \quad \therefore \; dz = \cos x \, dx,$

$$\cos^2 x = 1 - \sin^2 x = 1 - z^2.$$

$$\therefore \int \frac{dx}{\cos x} = \int \frac{dz}{1 - z^2} = \frac{1}{2} \log_\epsilon \frac{(1+z)}{(1-z)} = \frac{1}{2} \log_\epsilon \frac{1 + \sin x}{1 - \sin x}.$$

(See Example 6, page 115.)

The result obtained may be put into a simple form as follows :—

Remembering the general formula $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$ (page 248), we have

$$1 + \sin x = 1 + \cos \left(\frac{\pi}{2} - x \right) = 2 \cos^2 \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

Remembering the general formula $1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}$ (page 248), we have

$$1 - \sin x = 1 - \cos \left(\frac{\pi}{2} - x \right) = 2 \sin^2 \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

$$\therefore \int \frac{dx}{\cos x} = \log \left(\frac{1 + \sin x}{1 - \sin x} \right)^{\frac{1}{2}} = \log_\epsilon \cot \left(\frac{\pi}{4} - \frac{x}{2} \right).$$

In a similar manner it may be shown that $\int \frac{dx}{\sin x} = \log_\epsilon \tan \frac{x}{2}$.

Partial Fractions.—The student is familiar with the method by which, having given a number of fractions, these are brought to a common denominator, and the sum of the whole expressed by a single fraction. Thus, $\frac{1}{x-a} + \frac{1}{x+a} = \frac{2x}{x^2 - a^2}$. When a function to be integrated consists of a fraction, the numerator and denominator of which each consists of a series of integral powers of x, integration can often be effected by the converse process of decomposing the given fraction into a number of partial fractions.

Thus, let it be required to find the value of

$$\int \frac{x^4 + 2x^3 - 9x^2 - 13x + 3}{x^2 - x - 6} \, dx.$$

We must first divide the denominator of this fraction into the numerator, and so reduce the integral to the form

$$\int\left(x^2+3x+\frac{5x+3}{x^2-x-6}\right)dx=\int x^2dx+3\int xdx+\int\frac{5x+3}{x^2-x-6}dx.$$

The first two integrals on the right-hand side of this equation can be found directly. It should further be noticed that the highest power of x occurring in the numerator of the fractional expression, under the integral sign, to the extreme right, is smaller by unity than the highest power of x in the denominator.

Further, $x^2-x-6=(x-3)(x+2)$.

Let us assume that

$$\frac{5x+3}{(x-3)(x+2)}=\frac{\mathsf{A}}{x-3}+\frac{\mathsf{B}}{x+2} \quad . \quad . \quad . \quad . \quad (1).$$

It is required to find the values of A and B.

Multiplying both sides of (1) by $(x-3)(x+2)$, we get

$$5x+3=\mathsf{A}(x+2)+\mathsf{B}(x-3).$$

This equation must hold for all values of x. We may therefore, in turn, substitute such values of x as will make the coefficients of A and B vanish. Thus, substituting $x=3$, we get

$$(5\times 3)+3=\mathsf{A}(3+2)+\mathsf{B}(3-3).$$

$$18=5\mathsf{A}. \qquad \therefore\ \mathsf{A}=\frac{18}{5}.$$

Substituting $x=-2$, we get

$$(5\times -2)+3=\mathsf{A}(-2+2)+\mathsf{B}(-2-3).$$

$$\therefore\ -7=-5\mathsf{B}. \qquad \therefore\ \mathsf{B}=\frac{7}{5}.$$

$$\therefore\ \frac{5x+3}{x^2-x-6}=\frac{18}{5}\cdot\frac{1}{x-3}+\frac{7}{5}\cdot\frac{1}{x+2}.$$

Hence, finally,

$$\int\frac{x^4+2x^3-9x^2-13x+3}{x^2-x+6}\,dx$$

$$=\int x^2dx+3\int xdx+\frac{18}{5}\int\frac{dx}{x-3}+\frac{7}{5}\int\frac{dx}{x+2}$$

$$=\frac{x^3}{3}+\frac{3x^2}{2}+\frac{18}{5}\log_\epsilon(x-3)+\frac{7}{5}\log_\epsilon(x+2).$$

Example 1.—Find the value of $\int \frac{dx}{a^4 - x^4}$.

$$a^4 - x^4 = (a^2 + x^2)(a^2 - x^2) = (a^2 + x^2)(a - x)(a + x).$$

It should be noticed that $a^2 + x^2$ cannot be split up into factors without involving imaginary quantities. Thus, $a^2 + x^2 = x^2 - (-a^2) = (x - a\sqrt{-1})(x + a\sqrt{-1})$. Consequently, we will content ourselves with decomposing $a^4 - x^4$ into the three factors written down above, and writing the numerator of this partial fraction, of which the denominator is equal to $a^2 + x^2$, in the form $\mathsf{A}x + \mathsf{B}$. In this fraction, the highest power of x occurring in the numerator is less by unity than the highest power of x in the denominator.

Let $$\frac{1}{a^4 - x^4} = \frac{\mathsf{A}x + \mathsf{B}}{a^2 + x^2} + \frac{\mathsf{C}}{a - x} + \frac{\mathsf{D}}{a + x}.$$

Multiplying both sides by $(a^4 - x^4)$, we get

$$1 = (\mathsf{A}x + \mathsf{B})(a^2 - x^2) + \mathsf{C}(a^2 + x^2)(a + x) + \mathsf{D}(a^2 + x^2)(a - x).$$

In this fraction substitute $x^2 = -a^2$. $\therefore\ x = a\sqrt{-1}$.

$$1 = (\mathsf{A}a\sqrt{-1} + \mathsf{B})(a^2 - (-a^2)) + (\mathsf{C} \times 0) + (\mathsf{D} \times 0).$$

$$\therefore\ 1 = 2a^2(\mathsf{A}a\sqrt{-1} + \mathsf{B}).$$

The quantity to the right of this equation consists of the sum of two terms, of which one is real and the other is imaginary. These two terms are together equal to 1, a real quantity. But from the reasoning used in chapter v., it follows that a real quantity cannot be equal to the sum of a real and an imaginary quantity. Hence $\mathsf{A} = 0$.

$$\therefore\ \mathsf{B} = \frac{1}{2a^2}.$$

Now substitute $x = +a$.

$$1 = (\mathsf{A}a + \mathsf{B})(a^2 - a^2) + \mathsf{C}(a^2 + a^2)(a + a) + \mathsf{D} \times (a^2 + a^2)(a - a).$$

$$\therefore\ \mathsf{C} = \frac{1}{4a^3}.$$

Substituting $x = -a$, we get

$$1 = (\mathsf{A}a + \mathsf{B})(a^2 - a^2) + \mathsf{C} \times (a^2 + a^2)(a - a) + \mathsf{D}(a^2 + a^2)(a + a).$$

$$\therefore \ \mathsf{D} = \frac{1}{4a^3}.$$

$$\therefore \int \frac{dx}{a^4 - x^4} = \frac{1}{2a^2}\int \frac{dx}{a^2 + x^2} + \frac{1}{4a^3}\int \frac{dx}{a - x} + \frac{1}{4a^3}\int \frac{dx}{a + x}$$

$$= \frac{1}{2a^3}\tan^{-1}\frac{x}{a} - \frac{1}{4a^3}\log_e(a - x) + \frac{1}{4a^3}\log_e(a + x)$$

$$= \frac{1}{2a^3}\left(\tan^{-1}\frac{x}{a} + \frac{1}{2}\log\frac{a + x}{a - x}\right).$$

Example 2.—Find the value of $\int \frac{dx}{x^3 - 1}$.

$$(x^3 - 1) = (x - 1)(x^2 + x + 1).$$

Let
$$\frac{1}{x^3 - 1} = \frac{\mathsf{A}x + \mathsf{B}}{x^2 + x + 1} + \frac{\mathsf{C}}{x - 1}.$$

Multiply both sides by $(x^3 - 1)$.

$$1 = (\mathsf{A}x + \mathsf{B})(x - 1) + \mathsf{C}(x^2 + x + 1).$$

We can now proceed to substitute

$$x^2 + x + 1 = 0,$$

the corresponding value of x being equal to $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}\sqrt{-1}$. Following the principles explained in Example 1, page 124, the values of A, B, and C may be determined. This method is, however, rather tedious. The following method is to be preferred :—

We have

$$1 = (\mathsf{A} + \mathsf{C})x^2 + (\mathsf{B} + \mathsf{C} - \mathsf{A})x + (\mathsf{C} - \mathsf{A}).$$

With finite values assigned to the coefficients of x and x^2, this would be a quadratic equation, satisfied by two, and only two, values of x. But it must be satisfied for any value of x whatever, since the sum of the partial fractions must be equal to

$$\frac{1}{x^3 - 1}$$

for any value of x. Hence the two sides of the quadratic equation must be *identical*—that is,

$$\begin{aligned} C - B &= 1, \\ B + C - A &= 0, \\ A + C &= 0. \end{aligned}$$

Adding these three equations together, we get

$$3C = 1. \quad \therefore \; C = \frac{1}{3}.$$

From the first we have

$$B = C - 1 = \frac{1}{3} - 1 = -\frac{2}{3}.$$

From the third we have

$$A = -C = -\frac{1}{3}.$$

$$\begin{aligned} \int \frac{dx}{x^3 - 1} &= -\frac{1}{3}\int \frac{(x+2)dx}{x^2+x+1} + \frac{1}{3}\int \frac{dx}{x-1} \\ &= -\frac{1}{3}\int \frac{\left(\frac{2x+1}{2} + \frac{3}{2}\right)}{x^2+x+1}dx + \frac{1}{3}\int \frac{dx}{x-1} \\ &= -\frac{1}{6}\int \frac{(2x+1)dx}{x^2+x+1} - \frac{1}{2}\int \frac{dx}{x^2+x+1} + \frac{1}{3}\int \frac{dx}{x-1}. \end{aligned}$$

To effect the first integration, let $z = x^2 + x + 1$.

$$\therefore \; dz = (2x+1)dx.$$

$$\int \frac{(2x+1)dx}{x^2+x+1} = \int \frac{dz}{z} = \log_\epsilon z = \log_\epsilon(x^2+x+1).$$

The second integral is a special form of Example 7, case (a). (See page 116.)

$$\begin{aligned} \therefore \; \int \frac{dx}{x^2+x+1} &= \frac{2}{\sqrt{4-1}} \tan^{-1}\frac{2x+1}{\sqrt{4-1}} \\ &= \frac{2}{\sqrt{3}} \tan^{-1}\frac{2x+1}{\sqrt{3}}. \end{aligned}$$

The third integral is equal to $\log_e(x-1)$.

$$\therefore \int \frac{dx}{x^3-1} = -\frac{1}{6}\log_e(x^2+x+1) - \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{1}{3}\log_e(x-1).$$

Rearranging the terms, this result may be put in the simple form—

$$\int \frac{dx}{x^3-1} = \frac{1}{6}\log_e(x-1)^2 - \frac{1}{6}\log_e(x^2+x+1) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}$$
$$= \frac{1}{6}\log_e\frac{(x-1)^2}{x^2+x+1} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

The above examples should suffice to indicate the method of procedure when the function to be integrated is capable of being decomposed into partial fractions.

Exercises.—(1.) Find the value of $\displaystyle\int \frac{d\theta}{\sin^4\frac{\theta}{2}}$.

Answer. $-\frac{2}{3}\cot^3\frac{\theta}{2} - 2\cot\frac{\theta}{2}$.

(2.) Find the value of $\displaystyle\int \frac{d\theta}{\sin^3\frac{\theta}{2}}$.

Answer. $-\dfrac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \log_e\tan\frac{\theta}{4}$.

(3.) Prove that $\displaystyle\int \frac{dx}{a+b\cos x}$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\frac{\sqrt{(a-b)}\tan\frac{x}{2}}{\sqrt{(a+b)}} \text{ when } a > b,$$

and
$$\frac{1}{b^2-a^2}\cdot\frac{\sqrt{(b-a)}\tan\frac{x}{2}+\sqrt{(b+a)}}{\sqrt{(b-a)}\tan\frac{x}{2}-\sqrt{(b+a)}} \text{ when } b > a.$$

CHAPTER VIII.

APPLICATIONS TO GEOMETRICAL AND MECHANICAL PROBLEMS.

SOME applications of the integral calculus to geometrical problems have already been given in chapter vi., but our scope was there limited to those problems where only the standard integrals occurred. We shall now consider a number of problems in geometry and mechanics, many of which involve more difficult integrals.

GEOMETRICAL PROBLEMS.

Problem.—A circle is drawn with the origin as centre, and a radius equal to a. It is required to determine the area bounded by a part of the circle, two ordinates, and the axis of x.

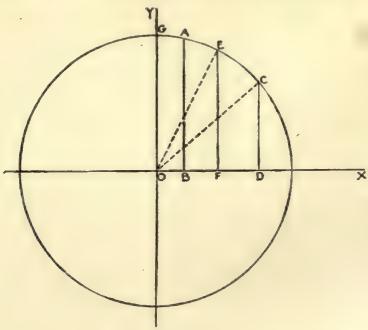

FIG. 34.

Referring to Fig. 34, let AB, CD be the two given ordinates; then it is required to determine the value of the area ACDB.

Let $OB = x_1$, $OD = x_2$.

If EF is any ordinate, then we have

$$(\mathsf{OF})^2+(\mathsf{FE})^2=\mathsf{OE}^2, \text{ or } x^2+y^2=a^2.$$
$$\therefore\ y=\sqrt{a^2-x^2}.$$

As proved in chapter vi., the area ACDB

$$=\int_{x_1}^{x_2} y\,dx=\int_{x_1}^{x_2}(a^2-x^2)^{\frac{1}{2}}dx.$$

From Example 4, page 119,

$$\int(a^2-x^2)^{\frac{1}{2}}dx=\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}.$$

$$\therefore\ \int_{x_1}^{x_2}(a^2-x^2)^{\frac{1}{2}}dx=\Big[\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big]_{x_1}^{x_2}.$$

As a particular instance, let $x_1=0$. Then the value of the definite integral is equal to the area OGCD.

$$\int_0^{x_2}(a^2-x^2)^{\frac{1}{2}}dx=\Big[\frac{x}{2}(a^2-x^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big]_0^{x_2}$$
$$=\frac{x_2}{2}(a^2-x_2{}^2)^{\frac{1}{2}}+\frac{a^2}{2}\sin^{-1}\frac{x_2}{a},$$

since both the first and second terms are reduced to zero when x is put equal to 0. ($\text{Sin}^{-1}0=$ the angle whose sine is equal to 0, $=0$.)

A geometrical interpretation may be given to the result just obtained. The area of a sector of a circle, such as GOC, is equal to the circular measure of the angle GOC, multiplied by half the square of the radius.

Now $\sin^{-1}\frac{x_2}{a}=$ the circular measure of the angle of which the sine is equal to $\frac{x_2}{a}\left(=\frac{\mathsf{OD}}{\mathsf{OC}}.\right)$ This angle is easily seen to be equal to $\angle$ OCD, which in its turn is equal to $\angle$ COG.

$$\therefore\ \frac{a^2}{2}\sin^{-1}\frac{x_2}{a}=\text{area of the sector COG.}$$

The area of the triangle COD is equal to

$$\mathsf{OD} \times \frac{\mathsf{DC}}{2} = x_2 \times \frac{y_2}{2} = x_2 \times \frac{(a^2 - x_2{}^2)^{\frac{1}{2}}}{2}.$$

Hence total area, GCDO,

$$= \frac{x_2}{2}(a^2 - x_2{}^2)^{\frac{1}{2}} + \frac{a^2}{2}\sin^{-1}\frac{x_2}{a}.$$

Problem.—To determine the length of an arc of the parabola $y^2 = 4ax$.

Let AOB (Fig. 32, page 107) represent the curve $y^2 = 4ax$. It is required to determine the length of the arc PB, cut off by the ordinates QP and CB.

Let $\mathsf{OQ} = x_1$, $\mathsf{OC} = x_2$, $\mathsf{OF} = x$, $\mathsf{FK} = \mathsf{E}h = dx$, $\mathsf{H}h = dy$.

Then the length of the element EH of the curve is obviously equal to

$$\left\{(dx)^2 + (dy)^2\right\}^{\frac{1}{2}} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx.$$

Also, since $\quad y = 2a^{\frac{1}{2}}x^{\frac{1}{2}},\ \dfrac{dy}{dx} = a^{\frac{1}{2}}x^{-\frac{1}{2}} = \sqrt{\dfrac{a}{x}}.$

$$\therefore \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} dx = \left(1 + \frac{a}{x}\right)^{\frac{1}{2}} dx = \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx.$$

To find the length of the arc PB of the parabola, we must integrate this quantity between the limits $x = x_1$ and $x = x_2$. Hence, length of arc PB

$$= \int_{x_1}^{x_2} \left(\frac{x+a}{x}\right)^{\frac{1}{2}} dx = \Big[\sqrt{(ax + x^2)} + a\log_e\{\sqrt{x} + \sqrt{x+a}\}\Big]_{x_1}^{x_2}$$

(See Example 8, page 121.)

Problem.—It is required to determine the area of the surface bounded by two radii vectores of a parabola (the pole being at the focus) and an arc of the curve.

Instead of defining the position of a point on a curve in terms of x and y, distances measured parallel to two rectangular axes, we may proceed as follows:—Take any fixed point (which we may call the *pole*), and draw a straight line from the pole to the given point on the curve. This straight line is termed a *radius vector.*

If the position and length of the radius vector are known, the position of the point on the curve becomes definitely known. Hence the position of a point on the curve may be defined in terms of the length r of the radius vector, and the angle θ which the latter makes with a fixed straight line.

r and θ are termed the *polar co-ordinates* of the point in question, and an equation between r and θ is termed the *polar equation* to the curve.

Let us now transform the equation to a parabola from Cartesian to polar co-ordinates.

Let the equation to the parabola AOB (Fig. 35), when referred to the rectangular axes OX and OY, be $y^2 = 4ax$.

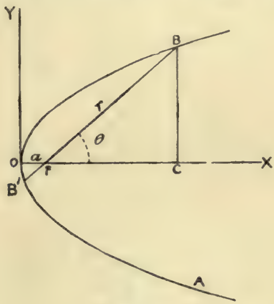

FIG. 35.

It is required to transform this equation into polar co-ordinates, the pole being the focus F, so that $OF = a$. (See page 21.)

Take any point B on the curve, and draw the radius vector $FB = r$, and also the ordinate $BC = y$. Let the angle $BFC = \theta$.

Further, $BC = y = BF \sin BFC = r \sin \theta$,

$$OC = x = OF + FC = OF + BF \cos BFC = a + r \cos \theta.$$

Hence, substituting these values of x and y in the equation $y^2 = 4ax$, we get

$$r^2 \sin^2\theta = 4a(a + r \cos \theta) = 4a^2 + 4ar \cos \theta.$$

This in itself may be considered to be the polar equation of the curve. It may, however, be put into a simpler form. It may be rewritten

$$r^2 \sin^2\theta - 4ar \cos \theta = 4a^2.$$

$$\therefore (r \sin \theta)^2 - 2 \cdot (r \sin \theta) \frac{2a \cos \theta}{\sin \theta} = 4a^2.$$

Complete the square on the left-hand side of this equation.

$$(r \sin \theta)^2 - 2 \cdot (r \sin \theta) \cdot \frac{2a \cos \theta}{\sin \theta} + \left(\frac{2a \cos \theta}{\sin \theta}\right)^2 = 4a^2 + \frac{4a^2 \cos^2\theta}{\sin^2\theta}$$
$$= 4a^2 \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta} = \frac{4a^2}{\sin^2\theta}.$$

Taking the square root of both sides, we get

$$r \sin \theta - \frac{2a \cos \theta}{\sin \theta} = \pm \frac{2a}{\sin \theta}. \qquad \therefore r = \frac{2a(\cos \theta \pm 1)}{\sin^2\theta}.$$

These two values of r, corresponding to the two roots of the above quadratic equation, give the lengths FB and FB′ (Fig. 35). Since we require the length FB, we must take the + sign. Hence, remembering that $1 + \cos \theta = 2 \cos^2 \frac{\theta}{2}$, and $\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$, we get

$$r = \frac{2a \cdot 2 \cos^2 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} = \frac{a}{\sin^2 \frac{\theta}{2}}.$$

This is the polar equation to the parabola, in which r is given as a function of θ. This result may be more simply obtained from first principles (see page 21). We have, focal distance of the point $B = r$. Perpendicular distance of B from directrix $= 2a + r \cos \theta$. Hence $r = 2a + r \cos \theta$.

$$\therefore \; r = \frac{2a}{1 - \cos \theta} = \frac{a}{\sin^2 \frac{\theta}{2}}.$$

We can now proceed to determine the area bounded by the two radii vectores FB, FA (Fig. 36), respectively making angles θ_1 and θ_2 with the line FN, which is the axis of symmetry of the parabola.

Draw any two radii vectores FC and FD, making a very small angle with each other. Let $\angle$ DFN $= \theta$, $\angle$ DFC $= d\theta$.

With F as centre, and FD as radius, describe the arc DE, and produce FC to E. Then, if FD $= r$, the length of the arc ED (which approximates closely to a straight line) is equal to $rd\theta$. We may treat FED as a triangle; the area of this will be equal to $\frac{r}{2} \times rd\theta = \frac{r^2 d\theta}{2}$. Now the area FED is equal to that of the element

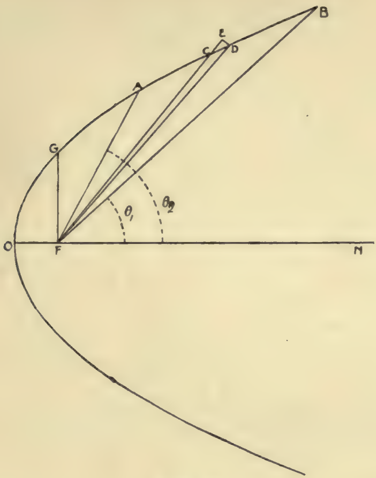

FIG. 36.

FCD of the curve, plus the area of the small triangle DCE. This latter quantity will be small in comparison with the area FCD, and may be neglected when $d\theta$ is indefinitely diminished. Hence area of element $\mathrm{FCD} = \dfrac{r^2 d\theta}{2}$. To find the area BFA, we must integrate between the limits $\theta = \theta_1$ and $\theta = \theta_2$.

$$\therefore \text{ Area BFA} = \int_{\theta_1}^{\theta_2} \frac{r^2}{2} d\theta = \frac{1}{2} \int_{\theta_1}^{\theta_2} \frac{a^2}{\sin^4 \frac{\theta}{2}} d\theta.$$

Now, remembering that $\cos^2 \dfrac{\theta}{2} + \sin^2 \dfrac{\theta}{2} = 1$, we have

$$\int \frac{d\theta}{\sin^4 \frac{\theta}{2}} = \int \frac{\left(\cos^2 \frac{\theta}{2} + \sin^2 \frac{\theta}{2}\right)}{\sin^4 \frac{\theta}{2}} d\theta = \int \cot^2 \frac{\theta}{2} \cdot \frac{1}{\sin^2 \frac{\theta}{2}} d\theta + \int \frac{d\theta}{\sin^2 \frac{\theta}{2}}.$$

Let $$z = \cot \frac{\theta}{2}.$$

Then $$dz = -\frac{1}{2}\frac{d\theta}{\sin^2\frac{\theta}{2}}.\quad \text{(See page 40.)}$$

$$\therefore \int \cot^2\frac{\theta}{2}\cdot\frac{1}{\sin^2\frac{\theta}{2}}\cdot d\theta = -2\int z^2 dz = -\frac{2z^3}{3} = -\frac{2}{3}\cot^3\frac{\theta}{2}.$$

$$\int \frac{d\theta}{\sin^2\frac{\theta}{2}} = -2\int dz = -2z = -2\cot\frac{\theta}{2}.$$

$$\therefore \frac{a^2}{2}\int_{\theta_1}^{\theta_2}\frac{d\theta}{\sin^4\frac{\theta}{2}} = \frac{a^2}{2}\Big[-\frac{2}{3}\cot^3\frac{\theta}{2} - 2\cot\frac{\theta}{2}\Big]_{\theta_1}^{\theta_2}.$$

As a particular instance, let $\theta_1 = \frac{\pi}{2}$, so that the corresponding value of r will be equal to half the *latus rectum* (see page 33); and let $\theta_2 = \pi$. We thus find the area of the surface FGO (Fig. 36). This is equal to

$$\frac{a^2}{2}\Big[-\frac{2}{3}\cot^3\frac{\theta}{2} - 2\cot\frac{\theta}{2}\Big]_{\frac{\pi}{2}}^{\pi} = \frac{a^2}{2}\left(\frac{2}{3}+2\right) = \frac{4a^2}{3}.$$

Since $\cot\frac{\pi}{4} = 1$, and $\cot\frac{\pi}{2} = 0$. This result agrees with that obtained on page 102, when x_2 is put equal to a, and $x_1 = 0$.

Problem.—To determine the area bounded by the two radii vectores corresponding to θ_1 and θ_2, and part of the logarithmic spiral, the polar equation of which is $r = a\epsilon^{b\theta}$.

The general form of a logarithmic spiral can be seen from Fig. 37. The angle θ is measured from the datum line PG. When $\theta = 0$, $r = a = $ PA. As θ increases, the value of r increases. In this curve θ is not confined within the limits 0 and 2π, but may go on increasing indefinitely. We thus get an indefinitely great number of loops, of which only the first two—namely, ABCD and DEFG—are shown in the figure.

Let $r_2 = $ PC and $r_1 = $ PB be the two radii vectores corresponding to the angles $\theta = $ CPA $= \theta_2$ and $\theta = $ BPA $= \theta_1$. Then, if $a = 1$,

$$\left.\begin{array}{l} r_2 = \mathrm{PC} = \epsilon^{b\theta_2} \\ r_1 = \mathrm{PB} = \epsilon^{b\theta_1}\end{array}\right\}. \qquad \begin{array}{l}\therefore \log_{10} r_2 = b\theta_2 \log_{10}\epsilon, \\ \log_{10} r_1 = b\theta_1 \log_{10}\epsilon.\end{array}$$

$$\therefore \frac{\log_{10} r_2}{\log_{10} r_1} = \frac{\theta_2}{\theta_1}.$$

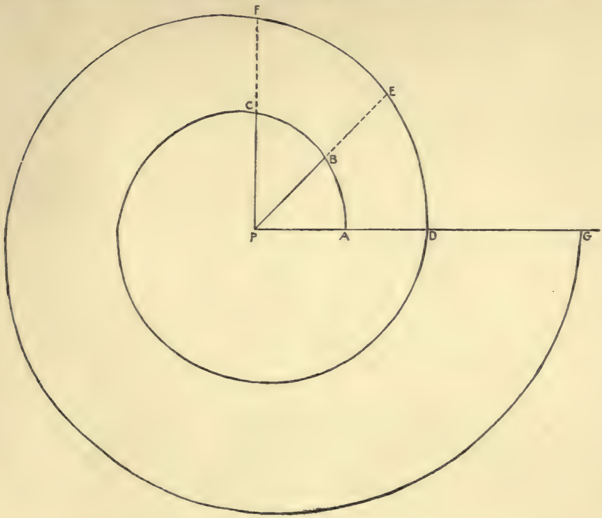

FIG. 37.

Hence the logarithm of a radius vector is proportional to the angle which has been swept out from the datum line PG. It is this property which has earned for the curve in question the title of logarithmic spiral.

In order to determine the area of the surface bounded by two radii vectores and part of the curve, we must find the value of

$$\int_{\theta_1}^{\theta_2} \frac{r^2}{2} d\theta.$$

But $r^2 = a^2 \epsilon^{2b\theta}$. Hence the area in question is given by

$$\frac{a^2}{2}\int_{\theta_1}^{\theta_2} \epsilon^{2b\theta} d\theta = \frac{a^2}{4b}\int_{\theta_1}^{\theta_2} 2b\epsilon^{2b\theta} d\theta = \frac{a^2}{4b}\Big[\ \epsilon^{2b\theta}\ \Big]_{\theta_1}^{\theta_2} = \frac{a^2}{4b}\left\{\epsilon^{2b\theta_2} - \epsilon^{2b\theta_1}\right\}.$$

Problem.—To find an expression for the length of an arc of a curve, the equation of which is expressed in terms of polar co-ordinates.

Let AB be the arc of the curve of which the length is required. Let PN be the datum line from which the angle θ is measured ;

so that, if $r_1 = \mathsf{PA}$, $r_2 = \mathsf{PB}$, are the radii vectores of the points A and B, $\angle \mathsf{APN} = \theta_1$, $\angle \mathsf{BPN} = \theta_2$.

Draw any two radii vectores PC, PD such that the angle CPD is very small. Let $\angle \mathsf{CPN} = \theta$, $\angle \mathsf{CPD} = d\theta$. With P as centre, and radius PC, describe the circular arc CE. Then, as $d\theta$ is

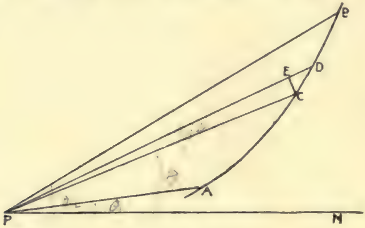

FIG. 38.

diminished indefinitely, DEC approximates more and more closely to a right-angled triangle, with the right angle at E.

$$\therefore\ (\mathsf{CD})^2 = (\mathsf{CE})^2 + (\mathsf{ED})^2.$$

CD represents a small increment of the arc of the curve AB.

Let $\mathsf{CD} = ds$. Also, since $r = \mathsf{PC} = \mathsf{PE}$, we may write $\mathsf{ED} = dr$. The length of CE is obviously equal to $rd\theta$.

$$\therefore\ (ds)^2 = (rd\theta)^2 + (dr)^2.$$

$$\therefore\ ds = \left((rd\theta)^2 + (dr)^2\right)^{\frac{1}{2}} = \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{1}{2}} d\theta.$$

To find the length of the arc AB, we must integrate this result between the limits $\theta = \theta_1$ and $\theta = \theta_2$.

$$\therefore\ \text{Length of AB} = \int_{\theta_1}^{\theta_2} \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\} d\theta.$$

Problem.—Find the length of an arc of the logarithmic spiral $r = a\epsilon^{b\theta}$.

Since $\dfrac{dr}{d\theta} = ab\epsilon^{b\theta}$, the required result

$$= \int_{\theta_1}^{\theta_2} (a^2\epsilon^{2b\theta} + a^2b^2\epsilon^{2b\theta})^{\frac{1}{2}} d\theta = a(1 + b^2)^{\frac{1}{2}} \int_{\theta_1}^{\theta_2} \epsilon^{b\theta} d\theta = \frac{a(1 + b^2)^{\frac{1}{2}}}{b} \Big[\epsilon^{b\theta} \Big]_{\theta_1}^{\theta_2}.$$

Problem.—Find the length of an arc of a parabola, using polar co-ordinates.

Let the pole be at the focus; then the equation to the parabola may be written

$$r = \frac{a}{\sin^2\frac{\theta}{2}}.$$

$$\therefore \frac{dr}{d\theta} = -\frac{2a}{\sin^3\frac{\theta}{2}} \times \cos\frac{\theta}{2} \times \frac{1}{2} = -\frac{a\cos\frac{\theta}{2}}{\sin^3\frac{\theta}{2}}.$$

$$\therefore \sqrt{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)} = \sqrt{\left(\frac{a^2}{\sin^4\frac{\theta}{2}} + \frac{a^2\cos^2\frac{\theta}{2}}{\sin^6\frac{\theta}{2}}\right)}$$

$$= a\sqrt{\left\{\frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{\sin^6\frac{\theta}{2}}\right\}} = a\sqrt{\left(\frac{1}{\sin^6\frac{\theta}{2}}\right)} = \frac{a}{\sin^3\frac{\theta}{2}}.$$

$\therefore$ The required length $\displaystyle = a\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin^3\frac{\theta}{2}}.$

But $$\frac{1}{\sin^3\frac{\theta}{2}} = \frac{2}{\sin\frac{\theta}{2}} \cdot \frac{\frac{1}{2}}{\sin^2\frac{\theta}{2}}.$$

Using the formula for integrating by parts (page 117), and putting $\displaystyle \frac{d\eta}{d\theta} = \frac{\frac{1}{2}}{\sin^2\frac{\theta}{2}}$, we find that $\eta = -\cot\frac{\theta}{2}$. (See page 40.)

Also, $$\xi = \frac{2}{\sin\frac{\theta}{2}}. \quad \therefore \frac{d\xi}{d\theta} = -\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}.$$

$$\therefore \int \frac{d\theta}{\sin^3\frac{\theta}{2}} = -2 \times \frac{\cot\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \int \frac{\cot\frac{\theta}{2} \cdot \cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} d\theta$$

$$= -\frac{2\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} - \int \frac{\cos^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} d\theta$$

$$= -\frac{2\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} - \int \frac{1-\sin^2\frac{\theta}{2}}{\sin^3\frac{\theta}{2}} d\theta$$

$$= -\frac{2\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} - \int \frac{d\theta}{\sin^3\frac{\theta}{2}} + \int \frac{d\theta}{\sin\frac{\theta}{2}}.$$

$$\therefore 2\int \frac{d\theta}{\sin^3\frac{\theta}{2}} = -\frac{2\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + 2\int \frac{\frac{1}{2}d\theta}{\sin\frac{\theta}{2}}.$$

The integral to the right of this equation is equal to $\log_e \tan\frac{\theta}{4}$ (see page 122). Hence, dividing through by 2, we get

$$\int \frac{d\theta}{\sin^3\frac{\theta}{2}} = -\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \log_e \tan\frac{\theta}{4}.$$

$\therefore$ Length of arc AB

$$= a\int_{\theta_1}^{\theta_2} \frac{d\theta}{\sin^3\frac{\theta}{2}} = a\left[-\frac{\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}} + \log_e \tan\frac{\theta}{4}\right]_{\theta_1}^{\theta_2}.$$

Centre of Gravity.

A fragment of matter is acted upon by gravity with a force proportional to its mass. If the fragment is very small, the point of application of the force may be left out of account, since all points within it must be very close together. In the case of a body of finite size, we must consider gravity to act separately on each individual element comprised in the body. There is always, however, a point in the body at which the application of a force of suitable magnitude will just equilibrate the action of gravity, and thus keep the body at rest. This point is termed the *centre of gravity of the body.* Since gravity pulls all elements of the body toward the surface of the earth, all the elements of force act on the body in the same direction; hence the single force which will just equilibrate these elements of force must be equal to the sum of the elements. When the unit of force used is that which gravity exerts on unit mass of matter, the equilibrating force will be equal in magnitude to the total mass of the body.

In a great number of mechanical problems it becomes necessary to determine the position of the centre of gravity of a body of some regular shape. A few of the most important cases will now be considered.

I. *Centre of gravity of two small, equal masses separated by a given distance.*

It can easily be seen that this lies on the line joining the two equal masses, at a point midway between them. For the symmetry of the arrangement affords no reason for locating the centre of gravity on one side of the line joining the two masses, rather than on the opposite side; hence it must lie on the line joining them. Further, there is no reason why the centre of gravity should be nearer to one mass than to the other, since both are equal in value; hence it must be midway between them.

II. *Centre of gravity of a thin, uniform straight rod.*

It can easily be seen that this must lie at the centre of the rod. For the rod may be imagined to be composed of a number of small elements of mass, situated symmetrically with regard to the centre of the rod. Thus the centre of gravity of each pair of equal elements, equidistant from the centre, and on opposite sides of it, will coincide with the centre of the rod.

III. *Centre of gravity of a thin, uniform circular ring.*

Imagine the ring to be divided into a number of small equal elements by means of numerous diameters of the circle. The centre of gravity of each pair of equal elements at opposite ends of a diameter will be at the centre of the circle; hence the centre of gravity of the whole will coincide with that point.

IV. *Since a circular lamina may be divided into a number of uniform concentric rings, each of which has its centre of gravity at the geometrical centre of the lamina, the centre of gravity of the whole lamina will coincide with its geometrical centre.*

The simple results obtained above will now be used to determine the centre of gravity of a number of solids of more complicated shapes.

Problem.—To find the centre of gravity of a triangular lamina.

Let m = the mass of unit area of the triangular lamina ABC (Fig. 39). Imagine the triangle to be divided into a number of

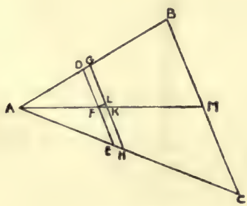

FIG. 39.

narrow strips, similar to DEHG, by lines parallel to BC. Then, since the centre of gravity of each of these strips will lie on the straight line AM, which passes through the centres of all of them, it is obvious that the centre of gravity of the triangular lamina will also lie on that line. Let $\bar{x}$ be the distance from A along AM at which the centre of gravity lies, and let $AM = b$. From F, the point of intersection of DE and AM, draw FL perpendicular to DE and GH. Then if the angle $KFL = a$, we have $\frac{FL}{FK} = \cos a$.

$\therefore FL = FK \cos a = dx \cos a$, if $AF = x$, $FK = dx$. The angle a will obviously be constant for all the strips.

Also, since the triangles ADE and ABC are similar,

$$\frac{DE}{AF} = \frac{BC}{AM} = k \text{ (say)}.$$

$$\therefore DE = kx.$$

Mass of element DEHG $= m \times$ area DEHG $= m \times$ DE $\times$ FL $= mkx \cos \alpha\, dx$, since DE and GH are indefinitely close together, and therefore indefinitely nearly equal in length.

Then, total mass of triangular lamina

$$= mk \cos \alpha \int_0^b x dx = mk \cos \alpha \cdot \frac{b^2}{2}.$$

Let us suppose that the line AM is placed horizontally, and the point A is supported. Then gravity acting on the strip DEHG will exert a force equal to $mkx \cos \alpha\, dx$, in a direction perpendicular to AM. The moment of this force about A is equal to $mkx \cos \alpha\, dx \times x = mk \cos \alpha\, x^2 dx$. The sum of the moments of the forces, acting on all the strips into which the lamina has been divided, must be equal to the moment about A of a force equal in magnitude to the mass of the lamina, and applied at a point on AM at a distance $\bar{x}$ from A. Hence, using the value of the mass of the lamina already found, and integrating the expression for the moment of the force acting on the strip DEHG, we get

$$\bar{x} \times mk \cos \alpha \frac{b^2}{2} = mk \cos \alpha \int_0^b x^2 dx = mk \cos \alpha \frac{b^3}{3}.$$

$$\therefore\ \bar{x} = \frac{2}{3} b.$$

The method used in this problem is essentially similar to that used in all cases.

Problem.—To find the centre of gravity of a right circular cone.

Let a be the vertical height of the cone, and let b be the radius of its circular base. Divide the cone into an indefinitely large number of parallel slices, by planes parallel to the base. Let x be the distance of one of these slices from the vertex, and let dx be its thickness. Then, if $r =$ radius of slice,

$$\frac{r}{x} = \frac{b}{a}. \quad \therefore\ r = x\frac{b}{a} = kx \text{ (say)}.$$

Volume of slice $= \pi r^2 \times dx = \pi k^2 x^2 dx.$

Then, if ρ be the density (mass of unit volume) of the cone, we have

$$\text{Mass of cone} = \rho \times \text{volume of cone} = \rho\pi k^2 \int_0^a x^2 dx = \rho\pi k^2 \frac{a^3}{3}.$$

Let us suppose that the cone is supported at its vertex, with the axis horizontal. Since the axis passes through the centre of each circular slice, and since the centre of gravity of each slice will lie on this axis, the centre of gravity of the whole cone will lie on it. Let its distance from the vertex be $\bar{x}$.

The moment, about the vertex of the cone, of the force exerted on the circular slice at a distance x from the vertex, is equal to

$$\rho \times \pi k^2 x^2 dx \times x = \rho\pi k^2 x^3 dx.$$

Hence, for reasons similar to those explained in the preceding problem,

$$\bar{x} \times \rho\pi k^2 \frac{a^3}{3} = \rho\pi k^2 \int_0^a x^3 dx = \rho\pi k^2 \frac{a^4}{4}.$$

$$\therefore \bar{x} = \frac{3}{4} a.$$

Problem.—Determine the centre of gravity of a uniform solid hemisphere.

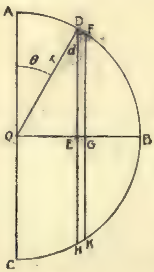

Fig. 40.

Let ABC (Fig. 40) be a section of the hemisphere, through the centre O, and perpendicular to the base. Let ρ be the density of the substance of which the hemisphere is composed. Then mass of hemisphere $= \frac{2\pi\rho}{3} r^3$, if $r =$ radius of hemisphere. (See p. 104.)

Draw OB through the centre O, and perpendicular to the base Divide the hemisphere into an indefinitely large number of slices, by planes parallel to the base. The centre of gravity of each slice will lie on OB.

Let DHKF (Fig. 40) be the section of one of these slices. Join O and D, and let $\angle \mathsf{DOA} = \theta$. Draw $\mathsf{F}d$ parallel to OB. Then, treating the arc DF as a straight line, we have, since

$$\angle \mathsf{ODF} = \frac{\pi}{2},$$

$$\angle \mathsf{FD}d + \angle \mathsf{EDO} = \angle \mathsf{EDO} + \angle \mathsf{DOE} = \angle \mathsf{EDO} + \left(\frac{\pi}{2} - \theta\right).$$

$$\therefore \angle \mathsf{FD}d = \left(\frac{\pi}{2} - \theta\right).$$

Length $\qquad \mathsf{DF} = r\,d\theta.$

Thickness of slice $= \mathsf{EG} = \mathsf{F}d = \mathsf{DF} \sin \mathsf{FD}d = r\,d\theta \times \sin\left(\frac{\pi}{2} - \theta\right)$
$= r \cos\theta\, d\theta.$

Radius of slice $= \mathsf{DE} = r \sin \mathsf{DOE} = r \cos\theta.$

Distance $\qquad \mathsf{OE} = r \cos \mathsf{DOE} = r \sin\theta.$

$\therefore$ Mass of slice $= \rho \times \pi \times (\mathsf{DE})^2 \times \mathsf{EG} = \rho \times \pi r^2 \cos^2\theta \times r \cos\theta\, d\theta$
$= \rho\pi r^3 \cos^3\theta\, d\theta.$

Moment, about O, of the force exerted by gravity on this slice $= \rho\pi r^3\cos^3\theta\, d\theta \times r \sin\theta = \rho\pi r^4 \cos^3\theta \sin\theta\, d\theta.$

Sum of moments of forces exerted by gravity on the whole of the slices comprised in the hemisphere

$$= \rho\pi r^4 \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta\, d\theta = \rho\pi r^4 \left[-\frac{\cos^4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\rho\pi r^4}{4}.$$

Hence, if $\bar{x}$ is the distance from O of the centre of gravity of the hemisphere, we have

$$\rho\,\frac{2\pi r^3}{3} \times \bar{x} = \frac{\rho\pi r^4}{4}. \quad \therefore \bar{x} = \frac{3}{8}r.$$

Problem.—Find the centre of gravity of a uniform semicircular lamina.

Let ABC (Fig. 40) now represent the semicircular lamina. If m = mass per unit area of the lamina, then mass of lamina $= m\,\frac{\pi r^2}{2}$.

Divide the lamina in strips, such as DHKF, by lines parallel to the diameter AC. Then, if $r = \text{OD}$, $\theta = \angle \text{AOD}$, the length DF $= rd\theta$, and breadth of strip $= \text{EG} = \text{DF} \sin \text{FD}d = rd\theta \times \cos\theta$. Also, $\text{DH} = 2\,\text{DE} = 2r \sin \text{DOE} = 2r\cos\theta$.

$\therefore$ Mass of strip DHKF $= m \times 2r\cos\theta \times rd\theta\cos\theta = 2mr^2\cos^2\theta\, d\theta$.

If the lamina is placed with OB horizontal, and is supported at O, moment of force exerted by gravity on strip DHKF about the point $\text{O} = 2mr^2\cos^2\theta\, d\theta \times r\cos \text{DOE} = 2mr^3\cos^2\theta \sin\theta\, d\theta$.

If the distance of the centre of gravity from O, along the line OB, is equal to $\bar{x}$, we have, as in previous examples,

$$m\frac{\pi r^2}{2} \times \bar{x} = 2mr^3\int_0^{\frac{\pi}{2}} \cos^2\theta \sin\theta\, d\theta = 2mr^3 \Big[-\frac{\cos^3\theta}{3}\Big]_0^{\frac{\pi}{2}} = \frac{2mr^3}{3}.$$

$$\therefore \bar{x} = \frac{4}{3\pi} r.$$

Problem.—Find the centre of gravity of a uniform hemispherical shell.

Let ABC (Fig. 40) represent a section of the hemispherical shell, of which OB is the axis. Let m be the mass per unit area of the shell. If DH, FK are sections of two planes, perpendicular to the axis OB, and very close to each other, these will cut off a circular strip from the shell; DF and HK will be sections of this strip. Hence, using the same notation as previously, breadth of strip $= \text{DF} = rd\theta$. Length of strip $= 2\pi\text{DE} = 2\pi\text{OD} \sin \text{DOE} = 2\pi r\cos\theta$. Mass of strip $= m \times 2\pi r\cos\theta \times rd\theta = 2\pi mr^2\cos\theta\, d\theta$. The centre of gravity of this circular strip will lie between E and G, on the axis OB; and when EG is indefinitely diminished, we may locate the centre of gravity of the strip at E.

$\therefore$ Moment of force, exerted by gravity on strip, about the point $\text{O} = 2\pi mr^2\cos\theta\, d\theta \times \text{OE} = 2\pi mr^3\cos\theta \sin\theta\, d\theta$. Also, mass of hemispherical shell $= 2\pi mr^2$. (See page 103.)

Hence, if the centre of gravity of the shell is on OB, at a distance $\bar{x}$ from O, we have

$$2\pi mr^2 \times \bar{x} = 2\pi mr^3\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta\, d\theta = 2\pi mr^3 \Big[\frac{\sin^2\theta}{2}\Big]_0^{\frac{\pi}{2}} = \pi mr^3.$$

$$\therefore \bar{x} = \frac{r}{2}.$$

Resultant Pressure and Centre of Pressure.

The pressure (force per unit area) at a point h centimetres below the surface of a liquid of density ρ, is equal to $h\rho$ grams per square centimetre. In the English system, $h\rho$ will give the pressure in pounds per square foot, if the density ρ is defined as the mass (in pounds) of a cubic foot of the liquid, and h is measured in feet.

The resultant force exerted by the superincumbent liquid, on a surface of given area, can be at once calculated by simple proportion, when the surface in question is horizontal, and all points on it are consequently at the same distance below the surface. On the other hand, when a plane surface is placed in a vertical plane below the surface of a liquid, different points on it will be at different distances below the surface, and therefore will be subjected to different pressures. In order to determine the resultant force exerted by the liquid on the surface, we must divide the surface into horizontal strips, find the general expression for the force exerted on one of these, and integrate the result between limits defined by the dimensions of the surface.

When a surface is placed in a definite position below the surface of a liquid, that point at which the resultant of all the forces exerted on different parts of one side of it may be considered to act is termed the *centre of pressure* of the surface.

The position of the centre of pressure of a surface is found by a method somewhat similar to that used in finding the centre of gravity of a body. This method will be made clear in the course of the solution of the following problems.

Problem.—Find the resultant horizontal pressure and the centre of pressure of a rectangular surface placed in a vertical plane, with one edge parallel to the surface of the liquid and at a distance h below it.

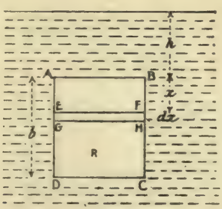

FIG. 41.

Let ABCD be the rectangular surface, placed in a vertical plane, with AB horizontal, and at a distance h below the surface of the liquid. Let $AB = a$, $BC = b$. Divide the surface into an indefinitely large number of narrow strips similar to EFHG, by means of lines parallel to AB. Let $AE = x$, $EG = dx$. Then, if the density of the liquid is ρ, the pressure (force per unit area) at a distance $(h+x)$ below its surface is equal to $(h+x)\rho$. Hence, the force exerted on $EFHG = \rho(h+x) \times$ area of $EFHG = \rho(h+x) \times adx$. Hence, the resultant force exerted on the plane

$$= \rho a \int_0^b (h+x)dx = \rho a\left(hb + \frac{b^2}{2}\right).$$

To find the centre of pressure (that is, the point where this resultant force may be considered to act), we must find a point such that the moment about the line AB of the resultant force applied at this point shall be equal to the sum of the moments of the forces exerted on the strips similar of EFHG about AB.

The moment about AB of the force exerted on EFHG $= \rho a(h+x)dx \times x = \rho a(h+x)xdx.$

It can easily be seen that the centre of pressure of the surface will lie on the vertical straight line, drawn midway between AD and BC. Let its distance below AB be denoted by $\bar{x}$.

$$\therefore \rho a\left(hb + \frac{b^2}{2}\right) \times \bar{x} = \rho a \int_0^b (h+x)xdx = \rho a\left(\frac{hb^2}{2} + \frac{b^3}{3}\right) x.$$

$$\therefore \bar{x} = \frac{\dfrac{hb^2}{2} + \dfrac{b^3}{3}}{hb + \dfrac{b^2}{2}} = \frac{\dfrac{hb}{2} + \dfrac{b^2}{3}}{h + \dfrac{b}{2}}.$$

If AB be in the surface of the liquid, $h = 0$, and we get

$$\bar{x} = \frac{2}{3} b.$$

If h be very large in comparison with b, we may write the result

$$\bar{x} = \frac{b\left(\dfrac{h}{2} + \dfrac{b}{3}\right)}{h + \dfrac{b}{2}} = \frac{\dfrac{bh}{2}}{h} = \frac{b}{2},$$

neglecting b in comparison with h. In this case the centre of pressure coincides with the centre of gravity of the rectangle.

Problem.—Find the centre of pressure of an equilateral triangle, immersed with one angle at a distance h below the surface of the liquid, and the opposite side horizontal and vertically beneath it.

Let ABC (Fig. 42) be the equilateral triangle, of which the

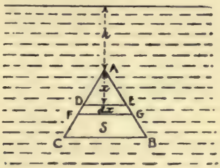

FIG. 42.

angle A is at a depth h below the surface, the side CB being horizontal, and vertically beneath A.

Let a = the length of a side of the triangle. Then the length of a perpendicular from A on to CB will be equal to $\frac{\sqrt{3}}{2}a$.

Divide the triangle into an indefinitely large number of narrow strips by means of lines parallel to CB. Let DEFG be one of these strips, and let x be its distance below A, and dx its breadth. Then

$$\frac{\mathrm{DE}}{x} = \frac{\mathrm{CB}}{\frac{\sqrt{3}}{2}a} = \frac{a}{\frac{\sqrt{3}}{2}a} = \frac{2}{\sqrt{3}}.$$

$$\therefore \ \mathrm{DE} = \frac{2}{\sqrt{3}}x.$$

Area of strip $= \frac{2}{\sqrt{3}}x \times dx.$

Resultant pressure on strip $= \rho(h+x) \times \frac{2}{\sqrt{3}}xdx.$

$\therefore$ Resultant pressure on triangle

$$= \frac{2}{\sqrt{3}} \cdot \rho \cdot \int_0^{\frac{\sqrt{3}}{2}a} (h+x)xdx$$

$$=\frac{2}{\sqrt{3}}\rho\left[\frac{hx^2}{2}+\frac{x^3}{3}\right]_0^{\frac{\sqrt{3}}{2}a}$$

$$=\frac{2}{\sqrt{3}}\rho\left\{\frac{3ha^2}{8}+\frac{\sqrt{3}\cdot a^3}{8}\right\}.$$

Moment about A of force exerted on strip DEGF

$$=\frac{2}{\sqrt{3}}\rho(h+x)xdx\times x=\frac{2}{\sqrt{3}}\rho(h+x)x^2dx.$$

If the centre of pressure lies at a distance $\bar{x}$ vertically beneath A, we have

$$\frac{2}{\sqrt{3}}\rho\left\{\frac{3ha^2}{8}+\frac{\sqrt{3}a^3}{8}\right\}\times\bar{x}=\frac{2}{\sqrt{3}}\rho\int_0^{\frac{\sqrt{3}}{2}a}(h+x)x^2dx$$

$$=\frac{2}{\sqrt{3}}\rho\left[\frac{hx^3}{3}+\frac{x^4}{4}\right]_0^{\frac{\sqrt{3}}{2}a}$$

$$=\frac{2}{\sqrt{3}}\rho\left\{\frac{\sqrt{3}ha^3}{8}+\frac{9a^4}{64}\right\}.$$

$$\therefore\ \bar{x}=\frac{\frac{\sqrt{3}ha^3}{8}+\frac{9a^4}{64}}{\frac{3ha^2}{8}+\frac{\sqrt{3}a^3}{8}}=\frac{\sqrt{3}\cdot ha+\frac{9a^2}{8}}{3h+\sqrt{3}a}.$$

If A be in the surface of the liquid, so that $h=0$, we get

$$\bar{x}=\frac{3\sqrt{3}}{8}a.$$

If h be great in comparison with a, we get

$$\bar{x}=a\frac{\sqrt{3}h+\frac{9a}{8}}{3h+\sqrt{3}a}=a\frac{\sqrt{3}h}{3h}=\frac{a}{\sqrt{3}};$$

that is, under these conditions the centre of pressure coincides with the centre of gravity of the triangle.

Problem.—Find the centre of pressure of a circle, immersed with a diameter vertical, and its highest point at a distance h beneath the surface of a liquid.

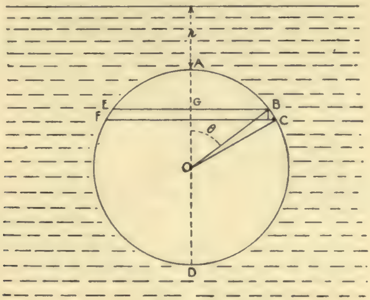

FIG. 43.

Let ABCDFE be the circle, with the diameter AD vertical, and its highest point A at a distance h beneath the surface of the liquid. As before, divide the circle into strips parallel to the surface; let EBCF be one of these. Join B and C to O, the centre of the circle, and let $OB = r$, $\angle BOA = \theta$, $\angle BOC = d\theta$. From B drop a perpendicular on to FC. Then, if $d\theta$ is small, we may treat the arc BC as a straight line, of which the length is $rd\theta$. Then, since $\angle OBC = \frac{\pi}{2}$, it can easily be seen that the perpendicular from B on to FC makes an angle $\left(\frac{\pi}{2} - \theta\right)$ with BC.

Breadth of strip $EBCF = BC \sin\theta = rd\theta \sin\theta$.
Length of strip $= 2 \times OB \sin BOA = 2r \sin\theta$.
$\therefore$ Area of strip $= 2r^2 \sin^2\theta \, d\theta$.
Also, $\qquad AG = OA - OG = r - r\cos\theta = r(1 - \cos\theta)$.

$\therefore$ Resultant force on strip

$$= \rho\{h + r(1 - \cos\theta)\} \times 2r^2 \sin^2\theta \, d\theta.$$

The resultant force on the whole of the circle is found by integrating the above result between the limits $\theta = 0$ and $\theta = \pi$.

$\therefore$ Resultant force on circle

$$= 2\rho r^2 \int_0^\pi (h + r - r\cos\theta)\sin^2\theta\, d\theta$$

$$= 2\rho r^2 \left\{(h+r)\int_0^\pi \sin^2\theta\, d\theta - r\int_0^\pi \sin^2\theta\cos\theta\, d\theta\right\}.$$

Also, since $\sin^2\theta = \dfrac{1 - \cos 2\theta}{2}$ (see page 247),

$$\int_0^\pi \sin^2\theta\, d\theta = \int_0^\pi \frac{1 - \cos 2\theta}{2} d\theta = \frac{1}{2}\int_0^\pi d\theta - \frac{1}{2}\int_0^\pi \cos 2\theta\, d\theta$$

$$= \frac{1}{2}\cdot {}_0^\pi\Big[\theta\Big] - \frac{1}{2}\,{}_0^\pi\left[\frac{\sin 2\theta}{2}\right]$$

$$= \left(\frac{\pi}{2} - \frac{0}{2}\right) - \frac{1}{2}\Big(0 - 0\Big) = \frac{\pi}{2}.$$

Further,

$$\int_0^\pi \sin^2\theta\cos\theta\, d\theta = {}_0^\pi\left[\frac{\sin^3\theta}{3}\right] = 0.$$

∴ Resultant pressure on circle

$$= 2\rho r^2 \times \frac{\pi}{2}(h + r) = \pi r^2 \rho (h + r).$$

This result shows that the resultant pressure is equal to the pressure at the centre of the circle multiplied by the area of the circle; in other words, the average pressure all over the circle is equal to that at its centre. This is obvious: for, from the symmetry of the circle, a horizontal strip at any distance above the centre will correspond with a similar strip at an equal distance below the centre; and the pressure at the upper strip will be as much less than that at the centre as the pressure at the lower strip will be greater than that at the centre.

Now the moment about A of the force exerted on the strip EBCF = force on EBCF × distance AG

$$= 2\rho r^2\{h + r(1 - \cos\theta)\}\sin^2\theta\, d\theta \times (r - r\cos\theta)$$
$$= 2\rho r^3\{h + r(1 - \cos\theta)\}(1 - \cos\theta)\sin^2\theta\, d\theta$$
$$= 2\rho r^3\{h + r - (h + 2r)\cos\theta + r\cos^2\theta\}\sin^2\theta\, d\theta.$$

The centre of pressure will obviously be on the vertical diameter AD. Let it be at a distance $\bar{x}$ from A. Then

$$\pi\rho r^2(h+r)\bar{x} = 2\rho r^3\int_0^\pi \{h+r-(h+2r)\cos\theta + r\cos^2\theta\}\sin^2\theta\,d\theta$$

$$= 2\rho r^3\left\{(h+r)\int_0^\pi \sin^2\theta\,d\theta - (h+2r)\int_0^\pi \sin^2\theta\cos\theta\,d\theta + r\int_0^\pi \cos^2\theta\sin^2\theta\,d\theta\right\}.$$

Now $\int_0^\pi \sin^2\theta\,d\theta = \frac{\pi}{2}$, as proved on page 150;

$\int_0^\pi \sin^2\theta\cos\theta\,d\theta = 0$, as proved on page 150;

$$\int \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{4}\int(2\sin\theta\cos\theta)^2 d\theta = \frac{1}{4}\int\sin^2 2\theta\,d\theta$$

$$= \frac{1}{4}\int\frac{1-\cos 4\theta}{2}\,d\theta.$$

$$\therefore \int_0^\pi \sin^2\theta\cos^2\theta\,d\theta = \frac{1}{8}\int_0^\pi (1-\cos 4\theta)d\theta = \frac{1}{8}\left\{\int_0^\pi d\theta - \int_0^\pi \cos 4\theta\,d\theta\right\}$$

$$= \frac{\pi}{8} - \left[\frac{\sin 4\theta}{4}\right]_0^\pi$$

$$= \frac{\pi}{8} - 0.$$

$$\therefore\ \pi\rho r^2(h+r)\bar{x} = 2\rho r^3\left\{\frac{(h+r)\pi}{2} - 0 + \frac{r\pi}{8}\right\} = \rho r^3\left\{(h+r)\pi + \frac{r\pi}{4}\right\}.$$

$$\therefore\ \bar{x} = r\left(1 + \frac{r}{4(r+h)}\right).$$

If $h = 0$ (that is, if the point A is in the surface of the liquid),

$$\bar{x} = \frac{5}{4}r.$$

If h is very great in comparison with r, we have

$$\bar{x} = r\left(1 + \frac{1}{4\left(1 + \frac{h}{r}\right)}\right) = r,$$

since $\frac{h}{r}$ will be a very large number, and consequently $\frac{1}{4\left(1 + \frac{h}{r}\right)}$ will be very small, and may be neglected.

Moments of Inertia.

When a particle of mass m is moving with a velocity v, the kinetic energy of the particle is equal to $\frac{1}{2}mv^2$. If the particle is revolving about an axis at a distance r from it, at the rate of n complete revolutions per second, the velocity v of the particle will be equal to $2\pi n \times r$. The quantity $2\pi n$ is termed the *angular velocity* of the particle; this is often denoted by ω.

Hence the kinetic energy of a particle of mass m, revolving about an axis, at a distance r, with an angular velocity ω, is equal to $\frac{1}{2}mr^2\omega^2$.

In the case of a rigid body of finite dimensions, rotating about an axis passing through the body, it is easily seen that the angular velocities of all particles comprised in the body will be equal. The kinetic energy of a particle of the body will, as previously explained, be equal to $\frac{1}{2}mr^2\omega^2$.

In order to find the kinetic energy of the rotating body, we must add together the kinetic energies of the elements into which we may suppose the body to be decomposed. Thus, if we imagine the body to be divided into elements of mass m_1, m_2, m_3, m_4 . . . . etc., at the respective distances r_1, r_2, r_3, r_4 . . . . etc., from the axis of rotation, then the kinetic energy of the body will be equal to

$$\begin{aligned}&\tfrac{1}{2}\{m_1r_1^2\omega^2 + m_2r_2^2\omega^2 + m_3r_3^2\omega^2 + m_4r_4^2\omega^2 + \ . \ . \ . \ .\}\\ &= \tfrac{1}{2}\{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + m_4r_4^2 + \ . \ . \ . \ .\}\omega^2 = \tfrac{1}{2}\Sigma mr^2\omega^2.\end{aligned}$$

Now, as long as the body rotates about a given axis, Σmr^2 will be constant. This quantity is termed the *moment of inertia* of the body about the given axis. If we write **I** for the moment of inertia of the body, then the kinetic energy of the body is equal to $\frac{1}{2}\mathbf{I}w^2$.

The student will have no difficulty in appreciating the importance of the function which we have termed the moment of inertia of a body. It must be used in all cases where the kinetic energy of a rotating body is required. In order to find the moment of inertia of a body about a certain axis, we must divide the body into a number of indefinitely small elements, obtain an expression for the product of the mass of each element into the square of its distance from the axis of rotation, and then find the sum of these expressions for all the elements of the body.

The moment of inertia of a thin circular ring about an axis through the centre, and at right angles to the plane of the ring, will obviously be equal to $\mathsf{M}r^2$, where M is the mass of the ring and r its radius; for each particle of the ring will be at the constant distance r from the axis of rotation. Similarly, the moment of inertia of a thin hollow cylinder of circular section, about the axis of the cylinder, will be equal to $\mathsf{M}r^2$, where M is its mass and r is the radius of its circular section.

We will now determine the moments of inertia of some solids of more complicated shape.

Problem.—Determine the moment of inertia of a thin rod, of mass M and length l, about an axis at right angles to the rod, and passing through its centre.

Let AB (Fig. 44) represent the rod, O being its middle point,

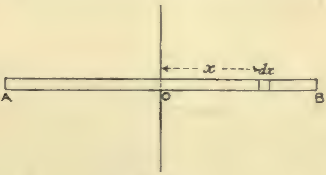

FIG. 44.

through which passes the axis about which the moment of inertia is to be determined. Divide the rod into short elements, and let dx be the length of one of these, at a distance x from O: x will be positive in the half OB of the rod, and negative in the half OA.

Let m be the mass of unit length of the rod, so that $ml = \mathsf{M}$.

Then mass of the element of length $dx = mdx$.

Distance of this element from the axis through $\mathsf{O} = x$.

$\therefore\ mdx \times x^2 =$ the element of the moment of inertia of the rod, due to the small length dx.

In order to obtain the moment of inertia of the rod, we must integrate this result between the limits $x = \frac{l}{2}$ and $x = -\frac{l}{2}$.

$\therefore$ Moment of inertia of rod

$$= m\int_{-\frac{l}{2}}^{+\frac{l}{2}} x^2 dx = m \left[\frac{x^3}{3}\right]_{-\frac{l}{2}}^{+\frac{l}{2}} = m\left\{\frac{1}{3}\left(\frac{l}{2}\right)^3 - \frac{1}{3}\left(-\frac{l}{2}\right)^3\right\}$$

$$= m\left(\frac{l^3}{24}+\frac{l^3}{24}\right) = \frac{ml^3}{12} = ml\,\frac{l^2}{12} = \frac{\mathsf{M}\,l^2}{12}.$$

Problem.—Determine the moment of inertia of a solid cylinder of circular section, about an axis coinciding with the geometrical axis of the cylinder.

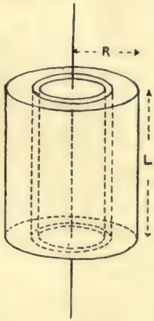

FIG. 45.

Let M be the mass, R the radius, and L the length of the cylinder. If ρ be the density of the substance of which the cylinder is composed, then $\mathsf{M} = \rho \times \pi \mathsf{R}^2 \mathsf{L}$.

Divide the cylinder into an indefinitely large number of thin-walled hollow tubes, the section of any one of these being circular, and concentric with the section of the solid cylinder. Let r be the distance from the axis of the cylinder to the inside of one of these tubular elements, and let dr be the thickness of its walls. Then the mass of the tubular element $= 2\pi r \times dr \times \mathsf{L} \times \rho = 2\pi\rho\mathsf{L}rdr$. Each particle composing the walls of this element will be at an approximate distance of r from the axis; therefore the moment of inertia of this element $= 2\pi\rho\mathsf{L}rdr \times r^2 = 2\pi\rho\mathsf{L}r^3dr$.

To determine the moment of inertia of the solid cylinder, we must integrate this quantity between the limits $r = 0$ and $r = \mathsf{R}$.

$\therefore$ Moment of inertia of solid cylinder

$$= 2\pi\rho\mathsf{L}\int_0^{\mathsf{R}} r^3dr = 2\pi\rho\mathsf{L}\,\frac{\mathsf{R}^4}{4} = \frac{\pi\rho\mathsf{L}\,\mathsf{R}^4}{2} = \rho \times \pi\mathsf{R}^2\mathsf{L}\cdot\frac{\mathsf{R}^2}{2} = \frac{\mathsf{M}\mathsf{R}^2}{2}.$$

Problem.—Determine the moment of inertia of a thick-walled hollow cylinder about its geometrical axis.

Let M be the mass, R_1 the internal radius and R_2 the external radius, and L the length of the cylinder.

Then, if ρ has the same meaning as in the last problem, we have $M = \rho\pi(R_2^2 - R_1^2)L$.

The form of the integral giving the moment of inertia of the hollow cylinder is similar to that obtained in the last problem. Here, however, the limits of integration are $r = R_1$ and $r = R_2$.

$\therefore$ Moment of inertia of hollow cylinder

$$= 2\pi\rho L \int_{R_1}^{R_2} r^3 dr = 2\pi\rho L \left[\frac{r^4}{4}\right]_{R_1}^{R_2}$$

$$= \pi\rho L\left(\frac{R_2^4 - R_1^4}{2}\right) = \rho\pi L\left(R_2^2 - R_1^2\right)\left(\frac{R_2^2 + R_1^2}{2}\right)$$

$$= M\,\frac{R_2^2 + R_1^2}{2}.$$

Problem.—Determine the moment of inertia of a thin circular lamina, about a diameter as axis.

Let $ABHFG$ (Fig. 46) be the circular lamina, and AF the axis

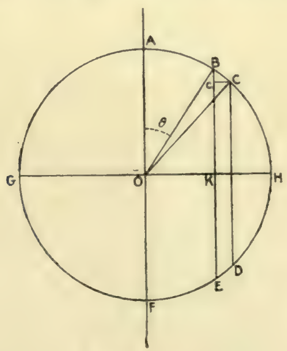

FIG. 46.

about which the moment of inertia is to be determined. Let O be the centre of the circle $ABEFG$, and let the radius $OA = r$, and let m = mass per unit area of the lamina.

If M be the total mass of the lamina, then $M = m \times \pi r^2$.

Divide the lamina into strips by means of numerous lines parallel to AF. Let BEDC be one of these strips. Join B and C to O, and let $\angle BOA = \theta$, $\angle BOC = d\theta$. Then the length of the arc BC is equal to $rd\theta$, and the breadth of the strip Cc is equal to BC $\sin$ CB$c = rd\theta \times \cos\theta$.

Also, length of strip

$$= BE = 2 \times OB \cos BOA = 2r \cos\theta.$$

Mass of strip

$$= m \times BE \times Cc = m \times 2r \cos\theta \times rd\theta \times \cos\theta = 2mr^2\cos^2\theta\, d\theta.$$

Distance of strip BCDE from axis AF

$$= OK = r \sin\theta.$$

∴ Increment of moment of inertia due to the strip BEDC = mass of strip $\times (OK)^2$

$$= 2mr^2\cos^2\theta\, d\theta \times r^2\sin^2\theta = 2mr^4\cos^2\theta \sin^2\theta\, d\theta.$$

If we integrate this expression between the limits 0 and $\frac{\pi}{2}$, we shall obtain that part of the moment of inertia due to the semicircle AHF; this quantity, multiplied by 2, will obviously give the moment of inertia of the whole lamina.

∴ Required moment of inertia

$$= 2 \times 2mr^4 \int_0^{\frac{\pi}{2}} \sin^2\theta \cos^2\theta\, d\theta$$

$$= mr^4 \int_0^{\frac{\pi}{2}} (2 \sin\theta \cos\theta)^2 d\theta = mr^4 \int_0^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta$$

$$= mr^4 \int_0^{\frac{\pi}{2}} \frac{1 - \cos 4\theta}{2}\, d\theta = \frac{mr^4}{2} \Big[\theta - \frac{\sin 4\theta}{4}\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{mr^4}{2}\frac{\pi}{2} = m \times \pi r^2 \times \frac{r^2}{4} = M\frac{r^2}{4}.$$

Problem.—Determine the moment of inertia of a solid cylinder about an axis passing at right angles through the middle point of the geometrical axis of the cylinder.

Let M be the mass of the cylinder, and ρ the density of the substance of which it is composed. Then, if l be the length of

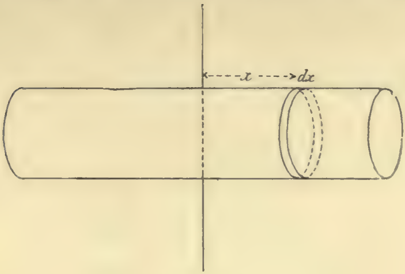

FIG. 47.

the cylinder, and r the radius of its circular section, we have $M = \rho \times \pi r^2 l$.

Let us suppose the cylinder to be divided into a number of circular discs, the thickness of one at a distance x from the axis, about which the moment of inertia is to be determined, being equal to dx. (See Fig. 47.)

Let us determine the kinetic energy due to this thin disc when the cylinder rotates at a rate of n turns per second.

It must first be noticed that whilst the cylinder performs one rotation, the disc revolves as a whole about the axis of rotation, and thus its centre of gravity describes a circle of radius r. In addition, the disc performs in the same time one complete rotation about a diameter as axis. The kinetic energy of the disc will be equal to the sum of its kinetic energies of revolution and rotation.

Mass of disc $= \rho\pi r^2 dx$.

Kinetic energy of disc due to its revolution about the axis

$$= \tfrac{1}{2}\rho\pi r^2 dx \times (2\pi n x)^2 = \tfrac{1}{2}\rho\pi r^2 dx \times (\omega x)^2 = \tfrac{1}{2} \cdot \rho\pi r^2 x^2 dx \cdot \omega^2.$$

Kinetic energy of disc due to its rotation about a diameter $= \frac{1}{2} \times$ moment of inertia of disc about a diameter $\times\, \omega^2$ (see p. 152)

$$= \tfrac{1}{2} \times \rho\pi r^2 dx \times \frac{r^2}{4} \times \omega^2.$$

Total kinetic energy of disc

$$= \tfrac{1}{2}\left\{\rho\pi r^2 x^2 dx + \frac{\rho\pi r^4 dx}{4}\right\}\omega^2.$$

$\therefore$ Increment of moment of inertia of cylinder due to the disc

$$= \rho\pi r^2\left(x^2 + \frac{r^2}{4}\right)dx.$$

If we integrate this expression between the limits $x=\frac{l}{2}$ and $x=-\frac{l}{2}$, we shall obtain the moment of inertia of the whole cylinder.

Moment of inertia of cylinder

$$=\rho\pi r^2\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left(\frac{r^2}{4}+x^2\right)dx$$

$$=\rho\pi r^2\left[\frac{r^2x}{4}+\frac{x^3}{3}\right]_{-\frac{l}{2}}^{+\frac{l}{2}}$$

$$=\rho\pi r^2\left(\frac{r^2l}{4}+\frac{l^3}{12}\right)=\rho\pi r^2l\cdot\left(\frac{r^2}{4}+\frac{l^2}{12}\right)$$

$$=\mathrm{M}\left(\frac{r^2}{4}+\frac{l^2}{12}\right).$$

When r is small in comparison with l, the solid cylinder approximates to a thin rod, and we get for its moment of inertia $\frac{\mathrm{M}l^2}{12}$, which agrees with the result obtained on page 154. When l is small in comparison with r, the cylinder approximates to a circular lamina, and we get for its moment of inertia $\mathrm{M}\frac{r^2}{4}$, which agrees with the result obtained on page 155.

Exercises.—(1.) Find the moment of inertia of a thin circular ring about a diameter.

Answer.—Mass of ring $\times\frac{(\text{radius})^2}{2}$.

(2.) Find the moment of inertia of a thin-walled hollow cylinder about a line drawn perpendicularly through the middle point of the geometrical axis.

Answer.—$\mathrm{M}\left(\frac{r^2}{2}+\frac{l^2}{12}\right)$.

(3.) Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Answer.—πab.

CHAPTER IX.

PHYSICAL PROBLEMS.

PROBLEM.—A hollow cylindrical vessel, of circular section, is partly filled with a liquid of density ρ, and is caused to rotate at a uniform speed about its geometrical axis, which is vertical. Determine the form of the free surface of the liquid.

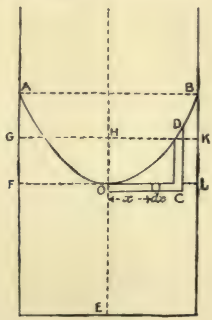

FIG. 48.

Let AOB (Fig. 48) be a section of the surface of the liquid, by a plane passing through the axis of the cylindrical vessel. Then, since the velocities of different parts of the liquid will vary only with the distance from the axis, and since the form of the surface is due to the differences in the velocities with which different parts of the liquid are moving, it follows that the section of the surface will be symmetrical with regard to the axis of the cylinder; in other words, the surface will be one of revolution.

Imagine a narrow tube, OCD (Fig. 48), to rotate uniformly with the cylinder and its contained liquid, and let the end O of the tube be immediately beneath the surface of the liquid at its lowest point—that is, on the axis of the cylinder—and let OC be horizontal and CD vertical, the sectional area of the tube being uniform throughout, and equal to a square centimetres. The presence of this tube will not in any way affect the form of the free surface of the liquid.

If the cylinder is rotating at a rate of n revolutions per second, a small element of the liquid in the tube, at a distance x from O, and of a length dx, will be moving with a velocity of $2\pi nx$ centimetres per second. The mass of the element will be equal to $\rho \times$ volume of element $= \rho a\, dx$ grammes.

Now, in order that a mass m should be constrained to move round a circle of radius r, with a uniform velocity v, it must be subjected to a pull, acting toward the centre of the circle, of a magnitude equal to $\frac{mv^2}{r}$. Since action and reaction are equal and opposite, we may say that a mass m, moving in the manner just prescribed, exerts a radial force, away from the centre, equal to $\frac{mv^2}{r}$. This is generally termed the centrifugal * force.

Hence the small element of liquid in the tube, of mass $\rho a\, dx$, and moving with a velocity $2\pi nx$ centimetres per second round a circle of radius x, will exert a force from O toward C of a magnitude $\frac{\rho a\, dx \times (2\pi nx)^2}{x} = \rho a(2\pi n)^2 x = \rho a\omega^2 x$, if $\omega = 2\pi n =$ the angular velocity of the vessel and its contained liquid (see page 152).

Each element of liquid in the tube will be exerting a force in the direction OC, and the total force in this direction at the point C will be equal to ap, the atmospheric pressure on the end O of the tube OC, *plus* the sum of the increments of force exerted by the various elements from O to C. If $\text{OC} = x$, we have

$$\text{Force at C in direction OC} = ap + \rho a\omega^2 \int_0^{x_1} x\,dx = a\left(p + \frac{\rho\omega^2 x_1{}^2}{2}\right).$$

But, for a liquid to be in equilibrium, the pressure in all directions must be equal. Hence, in the present case, the hydrostatic pressure over the bottom of the tube CD, due to the atmospheric pressure p, *plus* the head of liquid CD, must be equal to the radial force along OC. Let $g =$ the force in dynes exerted by gravity on

---

* Lat. *centrum*, the centre; and *fugio*, to flee from.

1 gramme of matter. Then hydrostatic pressure over bottom of tube $\mathsf{CD} = ap + g\rho a \times$ length $\mathsf{CD} = ap + g\rho a y_1$, if $\mathsf{CD} = y_1$.

$$\therefore\ a\left(p + \frac{\rho\omega^2 x_1^2}{2}\right) = a(p + g\rho y_1).$$

$$\therefore\ \frac{\omega^2 x_1^2}{2} = g y_1. \quad \therefore\ x_1^2 = \frac{2g}{\omega^2} y_1.$$

We have now determined a relation between the height $\mathsf{CD} = y_1$ of any portion of the surface of the liquid, and the distance x_1 of that portion of the surface from the axis of rotation.

$x^2 = \frac{2g}{\omega^2} y$ will thus be the equation to the surface AODB, the origin being taken at O, the lowest point on the surface. This equation is that of a parabola (see page 21), of which the focus is on the axis of rotation, at a distance $\frac{g}{2\omega^2}$ above O. Hence the free surface of the liquid has the form of a paraboloid of revolution. The form of the surface is independent of the density of the liquid and of the atmospheric pressure.

Problem.—Find the relation between the depression of the vertex of the paraboloid, below the level of the surface of the liquid when at rest, and the angular velocity ω.

Having determined the form of the free surface of the liquid, we must now employ the condition that the volume of the liquid remains constant at whatever speed the vessel may be rotating.

As proved on page 106, the volume enclosed by the paraboloid of revolution (of which AOB is an axial section) and the plane perpendicular to the axis (of which AB is a section) is equal to $\frac{1}{2}\mathsf{AF}\cdot\pi(\mathsf{OF})^2$ (Fig. 48) = one-half the volume of the cylinder, of which ABLR is the longitudinal section. Consequently, the liquid above the level FL will also occupy a volume equal to one-half of the volume of the cylinder, of which ABLF is a longitudinal section. Let GHK be the position of the surface of the liquid when at rest. Then, since the liquid occupying the volume $\pi(\mathsf{GH})^2 \times \mathsf{FG}$ when at rest, will occupy the volume $\frac{1}{2}\cdot\pi(\mathsf{OF})^2 \times \mathsf{AF} = \frac{1}{2}\pi(\mathsf{GH})^2 \times \mathsf{AF}$ when the liquid is revolving, we have the relation

$$\pi(\mathsf{GH})^2 \times \mathsf{FG} = \tfrac{1}{2}\pi(\mathsf{GH})^2 \times \mathsf{AF}.$$

$$\therefore\ \mathsf{FG} = \tfrac{1}{2}\mathsf{AF}.$$

Also, FG is the amount by which the vertex O of the paraboloid is depressed beneath the level of the liquid when at rest. Then, substituting OF for x in the equation to the parabola AOB, we get, for the corresponding value AF of y,

$$\mathsf{AF} = \frac{\omega^2}{2g} (\mathsf{OF})^2.$$

$$\therefore\ \mathsf{FG} = \tfrac{1}{2}\mathsf{AF} = \frac{(2\pi n)^2}{4g} (\mathsf{OF})^2 = \frac{\pi^2 n^2}{g} (\mathsf{OF})^2.$$

In this equation, OF is equal to the radius of the cylindrical vessel, and is therefore constant. Hence the depression of the vertex, beneath the level of the liquid when at rest, is proportional to the square of the angular velocity of the vessel, or to the square of the number of revolutions per second.

Problem.—If, in the last problem, the cylindrical vessel is closed above by a horizontal plate, and the angular velocity ω is sufficient for the liquid to reach this plate, then the distance of the vertex of the paraboloid below the plate is proportional to ω.

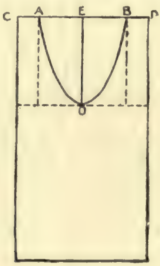

FIG. 49.

Let AOB (Fig. 49) represent the section of the free surface of the liquid, and let CD represent the section of the horizontal plate which closes the upper end of the cylinder. It is required to prove that the distance EO, from the horizontal plate CD to the vertex O of the paraboloid, is proportional to ω, the angular velocity of the vessel and its contained liquid.

As before, the equation to the parabola AOB will be given by $y = \frac{\omega^2}{2g} x^2$, where y is the distance of any point on the surface above the level of O, and x is the distance of that point from the axis.

The condition which defines the distance of O below E is that the volume of the liquid remains constant. Since the cylindrical vessel is closed, the volume unoccupied by the liquid must also remain constant—that is, the volume enclosed by the paraboloid of which AOB is a section, and the plane of which AB is a section, must be constant. This latter volume, according to the result obtained on page 106, is equal to

$$\tfrac{1}{2}\mathrm{EO} \times \pi(\mathrm{EA})^2.$$

$$\therefore\ \tfrac{1}{2}\mathrm{EO} \times \pi(\mathrm{EA})^2 = \text{a constant value, } k \text{ (say).}$$

But
$$\mathrm{EO} = \frac{\omega^2}{2g} \times (\mathrm{EA})^2. \quad \therefore\ (\mathrm{EA})^2 = \frac{2g}{\omega^2} \times \mathrm{EO}.$$

$$\therefore\ \frac{\pi}{2} \times \mathrm{EO} \times \frac{2g}{\omega^2}\mathrm{EO} = k.$$

$$\therefore\ \mathrm{EO}^2 = \frac{k}{\pi g}\omega^2, \text{ and}$$

$$\mathrm{EO} = \sqrt{\frac{k}{\pi g}} \cdot \omega = \sqrt{\frac{k}{\pi g}} \cdot 2\pi n.$$

Hence the distance EO of the vertex O of the paraboloid, below the plate closing the upper end of the cylinder, is directly proportional to ω, the angular velocity of the vessel, or to n, the number of its revolutions per second.

The result just obtained has been utilized in an ingenious manner by Mr. Killingworth Hedges in the construction of a fluid speed indicator. A glass cylinder, closed at both ends by plane discs, is filled with a mixture of glycerine and water to within a short distance of the upper end. When the cylinder is caused to rotate about a vertical axis agreeing with its geometrical axis, that part of the liquid which is farthest from the axis quickly rises until it comes in contact with the plate closing the upper end of the cylinder. After the speed necessary for this has been attained, the distance of the vertex of the paraboloid below the plate will be proportional to the speed of rotation of the cylinder, and an observation of this distance can thus be used to determine the number of its revolutions per second.

Problem.—Determine the work performed when a gas is compressed (*a*) under isothermal conditions, (*b*) under adiabatic conditions.

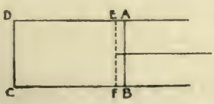

FIG. 50.

Let ABCD represent the section of a cylinder permanently closed at one end CD, and fitted with an air-tight and frictionless piston AB, which can be moved along the cylinder by the application of a sufficient force.

Let p be the pressure of the gas enclosed in the cylinder when the piston occupies the position AB, and let a be the area of the piston. Then the force resisting the inward motion of the piston $= pa$. If the piston be forced inwards to the position EF, through a distance $\text{EA} = dx$, the work done $= pa \times dx$. But $adx =$ the volume of the cylinder, of which EABF is the longitudinal section = the amount by which the volume of the gas has been diminished. Let v be the initial volume of the gas, then we may write $adx = -dv$, where dv indicates a small increment of volume of the gas. Then work performed on gas $= -pdv$.

Let us suppose that the connection between the pressure p and the volume v of the gas can be expressed by means of a curve, such as AB (Fig. 51). Here AC represents the initial pressure, and

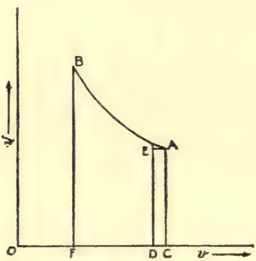

FIG. 51.

OC the initial volume of the gas, and $\text{DC} = -dv =$ the small amount by which the volume has been diminished. Then work performed in this small compression $= -pdv =$ area ACDE. If we compress the gas so that the curve AB is followed, then the total work done will consist of a number of elements similar to ACDE, and the sum of these elements will be equal to the area ABFC. The form of the curve AB will depend on the nature of the substance compressed, and also on the conditions under which the compression is effected.

(a) If the substance is a perfect gas, and the compression is performed isothermally (that is, under such conditions that its temperature remains constant), the equation to the curve AB will be—

$$pv = \text{a constant} = k \text{ (say)}.$$

$$\therefore p = \frac{k}{v}.$$

The work performed during the compression from A to B will be equal to

$$-\int_{v_1}^{v_2} \frac{k}{v} dv = -k \int_{v_1}^{v_2} \frac{dv}{v} = -k \Big[\log_\epsilon v \Big]_{v_1}^{v_2}$$

$$= -k \log_\epsilon \frac{v_2}{v_1} = k \log_\epsilon \frac{v_1}{v_2},$$

where $v_1 = \text{OC} =$ the initial volume of the gas,

$v_2 = \text{OF} =$ the final volume of the gas.

The value of k is equal to the product of the initial pressure (p_1, say) and the initial volume, v_1, of the gas. Also, since the Napierian logarithm of a quantity is equal to the logarithm to the base ten of that quantity, multiplied by 2·303 (page 95), we have the final result—

Work performed in compressing the gas isothermally = area ABFC (Fig. 51)

$$= 2{\cdot}303 \times p_1 v_1 \log_{10} \frac{v_1}{v_2}.$$

If we are given the initial and final pressures, p_1 and p_2, instead of the corresponding volumes v_1 and v_2, we have, since $p_1 v_1 = p_2 v_2$, and $\frac{v_1}{v_2} = \frac{p_2}{p_1}$,

Work performed during the isothermal compression

$$= 2{\cdot}303 \times p_1 v_1 \log_{10} \frac{p_2}{p_1}.$$

(*b*) If the substance is a perfect gas, and the compression is performed adiabatically (that is, under such conditions that heat is neither communicated nor abstracted by its surroundings), the equation to the curve AB will be *—

$$pv^\gamma = \text{constant} = \text{C (say)},$$

* See "Heat for Advanced Students," by the Author, page 317. (Macmillan and Co.)

where γ is the ratio of the constant pressure and constant volume specific heats of the gas. In the case of air, the value of γ is equal to 1·408.

Then $$p=\frac{\mathsf{C}}{v^\gamma},$$

and the work performed in passing along curve AB, the volume of the gas being thereby diminished from v_1 to v_2, is equal to

$$-\int_{v_1}^{v_2}\frac{\mathsf{C}dv}{v^\gamma}=-\mathsf{C}\left[\frac{v^{-\gamma+1}}{-\gamma+1}\right]_{v_1}^{v_2}=\frac{\mathsf{C}}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}}-\frac{1}{v_1^{\gamma-1}}\right),$$

where $$\mathsf{C}=p_1v_1^\gamma.$$

This result may also be expressed in a slightly different form as follows :—

Since $p_1v_1^\gamma=p_2v_2^\gamma=\mathsf{C}$, we have

$$\begin{aligned}\frac{\mathsf{C}}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}}-\frac{1}{v_1^{\gamma-1}}\right)&=\frac{p_1v_1^\gamma}{\gamma-1}\left(\frac{1}{v_2^{\gamma-1}}-\frac{1}{v_1^{\gamma-1}}\right)\\&=\frac{1}{\gamma-1}\left(\frac{p_1v_1^\gamma}{v_2^{\gamma-1}}-\frac{p_1v_1^\gamma}{v_1^{\gamma-1}}\right)\\&=\frac{1}{\gamma-1}\left(\frac{p_2v_2^\gamma}{v_2^{\gamma-1}}-\frac{p_1v_1^\gamma}{v_1^{\gamma-1}}\right)\\&=\frac{1}{\gamma-1}(p_2v_2-p_1v_1).\end{aligned}$$

This result brings into prominence a fact of some importance.

Let $pv=p_1v_1=\mathsf{RT}_1$ be the equation to the isothermal for $\mathsf{T}_1°$ (absolute).

Let $pv=p_2v_2=\mathsf{RT}_2$ be the equation to the isothermal for $\mathsf{T}_2°$ (absolute).

The above result may now be written in the simpler form—

Work performed during adiabatic compression $=\dfrac{\mathsf{R}}{\gamma-1}(\mathsf{T}_2-\mathsf{T}_1)$.

It at once becomes apparent that if a given quantity of gas is compressed adiabatically, so that its temperature rises from T_1 to T_2, the work performed will be independent of the initial pressure and volume of the gas. In other words, in passing along an adiabatic of a perfect gas from one isothermal to another, the work performed is the same, whatever adiabatic is traversed.

Problem.—Determine the entropy of 1 gramme of water at any temperature, the entropy of 1 gramme of water at 0° C. being taken as zero.

When a quantity of heat Q is communicated to a body, the temperature of the body meanwhile remaining constant at $T°$ (absolute), the entropy of the body is increased by $\frac{Q}{T}$.*

For instance, when a gramme of water at 100° C. is vaporized, a definite quantity of heat, termed the latent heat of water at 100°, is communicated to it, the temperature meanwhile remaining constant at 100° C. $(100 + 273 = 373°$ absolute).

Let L be the latent heat of water at 100°, then $\frac{L}{373}$ is the increase in the entropy of one gramme of the water substance during vaporization.

When water is heated, without the occurrence of vaporization, we can find the increase of entropy by considering the rise of temperature to occur in an indefinitely large number of stages, during each of which we may take the temperature as constant.

Thus, assuming that the specific heat of water is constant, and equal to unity, if we have a gramme of water at $T°$ (absolute), and raise its temperature to $(T + dT)°$, the quantity of heat communicated is equal to $1 \times dT = dT$. If dT is indefinitely small, $\frac{dT}{T}$ will differ by an infinitesimal amount from $\frac{dT}{T + dT}$; and since the increase of entropy will lie between these two quantities, we may write it down as equal to $\frac{dT}{T}$. In order to determine the entropy of a gramme of water at $T°$ (absolute), we must obtain the sum of the values of $\frac{dT}{T}$, while T varies from 273° absolute (corresponding to 0° C.) to T.

$$\therefore \text{ entropy of 1 gramme of water at } T° = \int_{273}^{T} \frac{dT}{T} = {}_{273}^{\quad T}\Big[\log_\epsilon T\Big]$$

$$= \log_\epsilon \frac{T}{273} = 2{\cdot}303 \log_{10} \frac{T}{273}.$$

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* See "Heat for Advanced Students," page 353.

Hence the equation connecting the entropy ϕ and the absolute temperature T of a gramme of water is given by

$$\phi = \log_e \frac{\mathsf{T}}{273} = 2 \cdot 303 \log_{10} \frac{\mathsf{T}}{273}.$$

Problem.—Determine the relation between the temperature T and the entropy ϕ of a gramme of saturated steam.

When a gramme of water at $t°$ C. is converted into saturated vapour *at the same temperature*, a quantity of heat L_t must be communicated to it.

According to Regnault,*

$$\mathsf{L}_t = 606 \cdot 5 - \cdot 695 t.$$

We have already obtained an expression for the entropy of a gramme of water at $\mathsf{T}°$ absolute. If we add to this the value obtained by dividing the value of L_t for the same temperature by the absolute temperature T, we shall obtain the entropy of a gramme of saturated steam at the absolute temperature T. Hence, remembering that $t°$ C. corresponds to $(\mathsf{T} - 273)$ on the absolute scale, we have—

Entropy ϕ of 1 gramme of saturated steam at $\mathsf{T}°$ absolute

$$= 2 \cdot 303 \log_{10} \frac{\mathsf{T}}{273} + \frac{606 \cdot 5 - \cdot 695(\mathsf{T} - 273)}{\mathsf{T}}.$$

Problem.—Determine the magnetic field at a point near to an infinitely long, straight conductor of negligible thickness, through which an electric current of C units (C.G.S.) is flowing.

In order to determine the magnetic field due to a given current flowing along any conductor, we must divide the conductor into short elements of length, determine the increment of the magnetic field due to the given current flowing through each element of the conductor, and, finally, obtain the sum of the increments of the magnetic field.

The following laws and definitions must be remembered :—

1. If a straight line be drawn at right angles through an element of a conductor along which an electric current is flowing, then at any point on this line the *magnitude* of the magnetic field due to that element varies directly as the product of the current and the length of the element, and inversely as the square of the dis-

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* See "Heat for Advanced Students," page 155. Temperature entropy diagrams are discussed on page 359.

tance of the point from the element. The *direction* of the field is perpendicular to the plane containing the element and the line on which the point is situate.

2. Let 1 centimetre length of wire be bent into an arc of a circle of 1 centimetre radius, so that each of the elements into which the wire may be divided is at a distance of 1 centimetre from the centre of the circle, and at right angles to the radius joining it to the centre, and let the magnetic field at the centre of the circle be equal to unity when a certain current flows along the wire, then that current is defined as having unit value.

Combining (1) and (2), we find that at a perpendicular distance r from an element of a conductor of length ds, along which a current C is flowing, the magnetic field due to that element is equal to $\frac{\mathrm{C}ds}{r^2}$.

3. A current flowing through an element of a conductor produces no magnetic field at any point in a line with the element.

4. At a point near a long conductor, along which a given current is flowing, the total magnetic field will be the same whether the conductor is straight, or bent into a number of small steps while following the same general course. In other words, a straight conductor can be replaced by one carrying an equal current, but bent into small steps, without altering the value of the magnetic field at a given distance from it.

We can now proceed to evaluate the magnetic field at a given distance from a long, straight conductor, carrying a given electric current C.

Let ABC (Fig. 52) be part of the given conductor, carrying a current C; and let it be required to determine the value of the magnetic field at a point D, at a perpendicular distance BD = R from the conductor. Divide the conductor into elements, such as ab (Fig. 52). Join a and b to D, and with D as centre, and Db as radius, describe the arc bc. If ab is very small, bc will approximate to a straight line, and acb will approximate to a right-angled triangle.

We may, in accordance with rule 4 above, replace the straight conductor AB, composed of elements similar to ab, by a conductor consisting of elements similar to bc, ca.

The point D will be a straight line with all the elements similar to ca. Hence, by rule 3 above, the current C flowing through these elements will not affect the magnetic field at D.

By rules 1 and 2, the magnetic field at D due to the current C

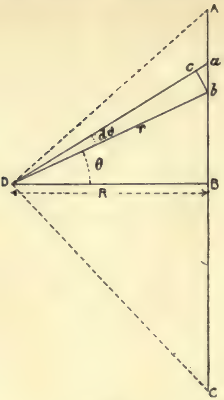

FIG. 52.

flowing through bc, will be perpendicular to the plane of the paper, and its magnitude will be

$$\frac{\mathsf{C} \times \text{length } cb}{(\mathsf{D}b)^2} = \frac{\mathsf{C} \times r d\theta}{r^2} = \mathsf{C}\frac{d\theta}{r},$$

if $$\mathsf{D}b = r,\ \angle b\mathsf{DB} = \theta,\ \angle b\mathsf{D}c = d\theta.$$

Let $\mathsf{DB} = \mathsf{R}$. Then

$$\frac{\mathsf{DB}}{\mathsf{D}b} = \frac{\mathsf{R}}{r} = \cos\theta.$$

$$\therefore \frac{1}{r} = \frac{\cos\theta}{\mathsf{R}}.$$

$\therefore$ magnetic field at D, due to current C flowing along the imaginary element bc, will be equal to

$$\frac{\mathsf{C}}{\mathsf{R}} \cdot \cos\theta\, d\theta.$$

This will also be equal to the magnetic field due to the current C flowing along the element ba of the conductor. To determine the total magnetic field due to the current C flowing along any finite part of the conductor, we must integrate this quantity between suitable limits. For instance, let it be required to determine the field due to the current C flowing along CA.

Join A and C to D, and let $\angle ADB=\theta_2$. $CDB=-\theta_1$. Then $\tan\theta_2=\frac{AB}{DB}$; $\tan\theta_1=\frac{BC}{DB}$.

The required magnetic field

$$=\frac{C}{R}\int_{-\theta_1}^{+\theta_2}\cos\theta\,d\theta=\frac{C}{R}\Big[\sin\theta\Big]_{-\theta_1}^{\theta_2}$$

$$=\frac{C}{R}\left\{\sin\theta_2-\sin(-\theta_1)\right\}=\frac{C}{R}\left\{\sin\theta_2+\sin\theta_1\right\}.$$

If the conductor extends to an infinite distance, both in the direction BA and the direction BC, we must write :—

$$\tan\theta_2=\frac{\infty}{DB}=\infty. \qquad \therefore\ \theta_2=\frac{\pi}{2}.$$

$$\tan\theta_1=\frac{\infty}{DB}=\infty. \qquad \therefore\ \theta_1=\frac{\pi}{2}.$$

Therefore, in this case, the magnetic field is equal in magnitude to

$$\frac{C}{R}\left(\sin\frac{\pi}{2}+\sin\frac{\pi}{2}\right)=\frac{2C}{R}.$$

Hence, at all points at a given distance from a linear conductor carrying a current C, the magnetic field will have the same value. It will always be at right angles to the plane containing the conductor AC and the line DB. Hence the magnetic lines of force will be circles concentric with the conductor.

Problem.—Determine the work that must be performed in moving a unit N pole once round a linear conductor carrying a current C.

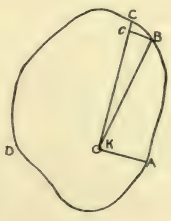

FIG. 53.

Let K represent the section of the linear conductor by a plane perpendicular to the conductor, and let the irregular figure ABCDA represent the path along which the unit N pole is to be carried.

Let us first determine the work performed in carrying the unit N pole along the short element BC of its path.

Join B and C to K, and with K as centre, and radius KB, describe the arc Bc. If CB is very short, CB and cB will approximate to straight lines, and CcB to a right-angled triangle.

$$\therefore \frac{c\text{B}}{\text{CB}} = \cos \text{CB}c = \cos \alpha \text{ (say)}. \qquad \therefore \text{CB} = \frac{c\text{B}}{\cos \alpha}.$$

Let distance BK $= r$, and $\angle$ BKC $= d\theta$. $\therefore$ CB $= \dfrac{rd\theta}{\cos \alpha}$.

Let the current flow downward through the paper, along the conductor of which K is a section. Then the magnetic field at B will be in the direction from c to B, and its magnitude will be equal to

$$\frac{2\text{C}}{r}.$$

In moving a unit N pole from B to C, the work done will be equal to the component magnetic field along CB, multiplied by the distance BC.

Component field along CB $=$ field along cB $\times \cos c$BC

$$= \frac{2\text{C}}{r} \times \cos \alpha.$$

$\therefore$ work performed in carrying a unit N pole from B to C

$$= \frac{2\text{C}}{r} \cos \alpha \times \frac{rd\theta}{\cos \alpha} = 2\text{C}d\theta.$$

In passing right round the curve ABCDA, we must perform an amount of work equal to the sum of an indefinitely large number of terms, each of which is equal to $2\text{C}d\theta$; and if we measure θ from the line KA, θ must vary from 0 to 2π.

$\therefore$ total work performed in carrying unit N pole once round the conductor carrying a current C

$$= 2\text{C}\int_0^{2\pi} d\theta = 4\pi\text{C}.$$

This result is one of great importance. It shows that the work performed while encircling the current will be the same whatever path is chosen. If the path chosen is near to the conductor, the magnetic field is great, but the distance traversed is small. If the path is remote from the conductor, the magnetic field is small, but the distance traversed is great.

Exercises.—(1.) Prove that, at a perpendicular distance R from an infinitely long cylindrical conductor charged with σ electrostatic units of electricity per unit length, the electrostatic field will be equal to $\dfrac{2\sigma}{\mathsf{R}}$.

(2.) Prove that, if a unit N pole be brought from an infinite distance along the axis to the centre of a circular turn of wire carrying a current of C units (C.G.S.), then the work performed is equal to $2\pi\mathsf{C}$ ergs.

Suggestion.—Let r = radius of circular turn of wire. Magnetic field at distance x from the centre along the axis

$$= \frac{2\pi \mathsf{C} r^2}{(r^2+x^2)^{\frac{3}{2}}}.$$

(Compare argument used on pages 176, 177.)

Let the motion of the N pole be opposed by this field. Then the work done under above conditions

$$= 2\pi \mathsf{C} r^2 \int_{\infty}^{0} \frac{-dx}{(r^2+x^2)^{\frac{3}{2}}}.$$

(Negative sign, because x is decreasing.)

To evaluate this integral, put $x = r \tan\theta$.

Then $$dx = \frac{r}{\cos^2\theta}\,d\theta,$$

$$r^2 + x^2 = r^2(1+\tan^2\theta) = \frac{r^2}{\cos^2\theta}.$$

When $x = r\tan\theta = \infty$, $\tan\theta = \infty$, and $\theta = \dfrac{\pi}{2}$.

When $x = r\tan\theta = 0$, $\tan\theta = 0$, and $\theta = 0$.

$$\therefore \text{ Work done } = -2\pi\mathsf{C}\int_{\frac{\pi}{2}}^{0} \frac{\dfrac{r^3 d\theta}{\cos^2\theta}}{\dfrac{r^3}{\cos^3\theta}} = -2\pi\mathsf{C}\int_{\frac{\pi}{2}}^{0} \cos\theta\, d\theta = 2\pi\mathsf{C}.$$

CHAPTER X.

DOUBLE AND TRIPLE INTEGRATION.

LET it be required to determine the area bounded by the two curves AB, CD (Fig. 54), and the ordinates AC and BD. The most simple method of effecting this result has already been explained. Another method, which leads to an identical result, will now be considered, in order to illustrate the nature of, and general method of dealing with, a double integral.

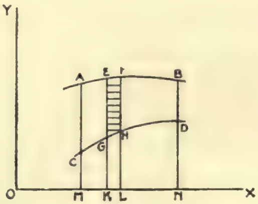

FIG. 54.

Divide the area ABDC into a number of parallel strips, similar to EFHG, by means of ordinates, and then subdivide each strip into a number of small rectangles, by means of lines parallel to the axis of x. Then, if $KL = dx$, and if the distance between two successive horizontal lines is equal to dy, the area of the element bounded by these lines will be equal to $dx \times dy$.

If we integrate this quantity with respect to y between the limits $y = y_1 = KG$ and $y = y_2 = KE$, treating dx as a constant, we shall obtain the area of the strip EFHG.

Thus, area EFHG

$$= \int_{y_1}^{y_2} dxdy = dx \int_{y_1}^{y_2} dy = dx(y_2 - y_1).$$

If we now integrate this result with respect to x between the limits $x = x_1 = \text{OM}$ and $x = x_2 = \text{ON}$, we shall obtain the sum of the areas of the strips, similar to EFHG, into which the surface ABDC has been divided. It must be noticed that y_2 and y_1 will have different values as we pass from strip to strip; in other words, we must express y_2 and y_1 as functions of x. If the equation to the curve AB is $y = \phi(x)$, and the equation to the curve CD is $y = \psi(x)$, then we may write down, for the area ABDC, this expression—

$$\int_{x_1}^{x_2}\int_{\psi(x)}^{\phi(x)} dxdy = \int_{x_1}^{x_2}\{\phi(x) - \psi(x)\}dx.$$

The quantity on the left-hand side of the equation is termed a double integral, since its evaluation involves two separate integrations. In the first operation, dy is integrated between the limits $y = \phi(x)$ and $y = \psi(x)$, the quantity dx being treated as a constant, since EG and FH are parallel. In the second operation, the quantity $\{\phi(x) - \psi(x)\}dx$, which is equal to the area of the strip EFHG, is integrated between the limits $x = x_1 = \text{OM}$ and $x = x_2 = \text{ON}$.

As explained on page 97, the integral

$$\int_{x_1}^{x_2}\phi(x)dx$$

is equal to the area ABNM (Fig. 54). Similarly, the integral

$$\int_{x_1}^{x_2}\psi(x)dx$$

is equal to the area CDNM. Hence the area of ABDC is obviously equal to

$$\int_{x_1}^{x_2}\phi(x)dx - \int_{x_1}^{x_2}\psi(x)dx = \int_{x_1}^{x_2}\{\phi(x) - \psi(x)\}dx.$$

Hence it is not generally necessary to perform a double integration in evaluating an area. The foregoing must be considered merely as an introduction to the meaning and method of dealing with a double integral.

A number of problems which cannot conveniently be solved without employing double integrals will now be considered.

Problem.—A circular channel of rectangular section is turned in the edge of a wooden disc, and this channel is filled with a large number of turns of insulated wire. It is required to determine the magnetic field at a point on the axis of the coil thus formed when a given current flows through the wire.

Let ABCD and EFGH (Fig. 55) be sections of the coil, by

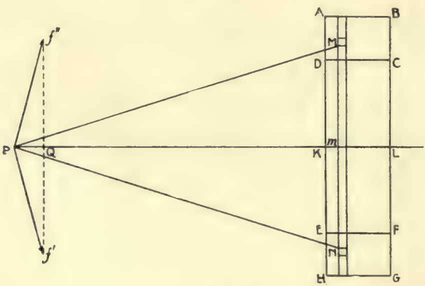

FIG. 55.

a plane passing through its axis PL. Let there be n wires passing through unit area of ABCD or EFGH, and let C be the current (in C.G.S. units) flowing along any wire. Let P be the point on the axis of the coil at which the magnetic field is to be determined.

Take P as origin, and measure x along PL, and y at right angles to PL.

Divide the surfaces ABCD and EFGH into strips parallel to the axis of y, by means of straight lines at a distance dx apart, and subdivide these strips into small rectangles by means of straight lines, parallel to the axis of x, at a distance dy apart. Let M be one of these elementary rectangles; the area of this will be equal to $dx \times dy$, and the number of wires passing through it will be equal to $n \cdot dx \cdot dy$. Let the distance between P and this rectangle (namely, PM) be equal to r, and let Pm be denoted by x, and mM by y. Then $r = (x^2 + y^2)^{\frac{1}{2}}$.

Consider the magnetic field at P due to the current C flowing along a short length dl of the $ndxdy$ wires passing through M. This field will be at right angles to PM, in the direction Pf', if the current flows *upwards* through M. Its magnitude will be equal to $\frac{\mathsf{C}dl \times ndxdy}{r^2}$ (see page 169). Let $\mathsf{P}f'$ be equal to this value. If N be the elementary rectangle in the area EFGH, which corresponds to M in ABCD, then the current will flow *downwards* through N, and the magnetic field due to this current, flowing along a length dl of the wires, will be equal to $\mathsf{P}f''$, at right angles to PN. Hence the components of $\mathsf{P}f'$ and $\mathsf{P}f''$ at right angles to PL will just neutralize each other. Consequently the effective component of the magnetic field due to the current C flowing upward along the $ndxdy$ wires, of length dl, passing through M, will be equal to PQ.

But since the triangles $\mathsf{P}f'\mathsf{Q}$ and $\mathsf{M}m\mathsf{P}$ are similar, we have

$$\frac{\mathsf{PQ}}{\mathsf{P}f'}=\frac{\mathsf{M}m}{\mathsf{PM}}=\frac{y}{r}. \quad \therefore \mathsf{PQ}=\mathsf{P}f' \times \frac{y}{r}=\frac{\mathsf{C}dl \cdot ndxdy}{r^2}\cdot\frac{y}{r}.$$

Each element dl of the circular ring, of which M and N are sections, will be at the same distance from P, and the component magnetic force along PL will have the value just found. Since the total length of this ring will be equal to $2\pi y$, the resultant magnetic field due to the current C flowing along the $ndxdy$ wires of which the ring is composed will have the value

$$2\pi y\mathsf{C} \times \frac{ndxdy}{r^2} \times \frac{y}{r}=\frac{2\pi \mathsf{C} ny^2dxdy}{r^3}=\frac{2\pi \mathsf{C} ny^2dxdy}{(x^2+y^2)^{\frac{3}{2}}}.$$

To obtain the total magnetic field at P, we must integrate the above expressions over the surface ABCD, since by this means we sum up the components of the magnetic field due to the elementary rings into which the coil has been decomposed. The limits of the integration are $x=\mathsf{PK}=x_1$, $x=\mathsf{PL}=x_2$; and $y=\mathsf{KD}=y_1$, $y=\mathsf{KA}=y_2$.

It is most convenient to integrate, first with respect to x, and then with respect to y. In the first operation we determine the component magnetic field at P, due to the current C flowing through all the wires at distances between y and $y+dy$ from the axis PL. In the second operation we sum up the component fields due to the current C flowing through all the wires between the distances $y=y_1$ and $y=y_2$ from the axis.

$$\text{Total field at } \mathsf{P}=2\pi\mathsf{C}n\int_{y_1}^{y_2}\int_{x_1}^{x_2}\frac{y^2dydx}{(y^2+x^2)^{\frac{3}{2}}}.$$

In the integration with respect to x, we must treat y and dy as constants. Now,

$$\int \frac{y^2dx}{(y^2+x^2)^{\frac{3}{2}}} = \int \frac{(y^2+x^2-x^2)dx}{(y^2+x^2)^{\frac{3}{2}}} = \int \frac{dx}{(y^2+x^2)^{\frac{1}{2}}} - \int \frac{x^2dx}{(y^2+x^2)^{\frac{3}{2}}}.$$

Integrating by parts, we have

$$\int \frac{x^2dx}{(y^2+x^2)^{\frac{3}{2}}} = \int x \cdot \frac{xdx}{(y^2+x^2)^{\frac{3}{2}}} = -x \cdot \frac{1}{(y^2+x^2)^{\frac{1}{2}}} + \int \frac{dx}{(y^2+x^2)^{\frac{1}{2}}}.$$

$$\therefore \int \frac{y^2dx}{(y^2+x^2)^{\frac{3}{2}}} = \int \frac{dx}{(y^2+x^2)^{\frac{1}{2}}} + x \cdot \frac{1}{(y^2+x^2)^{\frac{1}{2}}} - \int \frac{dx}{(y^2+x^2)^{\frac{1}{2}}}$$

$$= \frac{x}{(y^2+x^2)^{\frac{1}{2}}}.$$

$$\therefore \int_{x_1}^{x_2} \frac{y^2dx}{(y^2+x^2)^{\frac{3}{2}}} = \frac{x_2}{(y^2+x_2^2)^{\frac{1}{2}}} - \frac{x_1}{(y^2+x_1^2)^{\frac{1}{2}}}.$$ (See also p. 173.)

$$\therefore 2\pi \mathrm{C} n \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{y^2dydx}{(y^2+x^2)^{\frac{3}{2}}} = 2\pi \mathrm{C} n \int_{y_1}^{y_2} \left\{ \frac{x_2}{(y^2+x_2^2)^{\frac{1}{2}}} - \frac{x_1}{(y^2+x_1^2)^{\frac{1}{2}}} \right\} dy$$

$$= 2\pi \mathrm{C} n \left\{ x_2 \int_{y_1}^{y_2} \frac{dy}{(x_2^2+y^2)^{\frac{1}{2}}} - x_1 \int_{y_1}^{y_2} \frac{dy}{(x_1^2+y^2)^{\frac{1}{2}}} \right\}$$

$$= 2\pi \mathrm{C} n \left\{ x_2 \Big[\log_\epsilon (y + \sqrt{x_2^2+y^2}) \Big]_{y_1}^{y_2} - x_1 \Big[\log_\epsilon (y + \sqrt{x_1^2+y^2}) \Big]_{y_1}^{y_2} \right\}$$

$$= 2\pi \mathrm{C} n \left\{ x_2 \log_\epsilon \frac{y_2 + \sqrt{x_2^2+y_2^2}}{y_1 + \sqrt{x_2^2+y_1^2}} - x_1 \log_\epsilon \frac{y_2 + \sqrt{x_1^2+y_2^2}}{y_1 + \sqrt{x_1^2+y_1^2}} \right\}.$$

Corollary.—Let a uniform thin layer of wire be wound on a cylinder of infinite length, and let it be required to determine the magnetic field at a point on the axis of the cylinder. Then we may dispense with the integration with respect to y, writing

$ndy = \mathsf{N} =$ number of turns per unit length of the cylinder. The required magnetic field is equal to

$$2\pi\mathsf{CN}\int_{-\infty}^{+\infty}\frac{y^2dx}{(y^2+x^2)^{\frac{3}{2}}} = 2\pi\mathsf{CN}\left[-\frac{x}{(y^2+x^2)^{\frac{1}{2}}}\right]_{x=-\infty}^{x=+\infty}$$

$$= 2\pi\mathsf{CN}\times 2\times\left[\frac{1}{\left(1+\left(\frac{y}{x}\right)^2\right)^{\frac{1}{2}}}\right]_{x=0}^{x=+\infty} = 2\pi\mathsf{CN}\times 2\times(1-0) = 4\pi\mathsf{CN}.$$

Problem.—Determine the magnitude of the magnetic field at a point (1) outside and (2) inside an infinitely long cylindrical conductor of finite diameter, along which an electrical current is flowing.

Let ABCD (Fig. 56) represent the section of the cylindrical

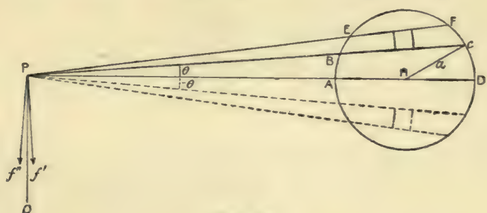

FIG. 56.

conductor of finite diameter AD, this conductor being supposed to extend indefinitely both above and below the paper. Let P be the point outside the conductor at which the field is required.

Draw any two lines PBC, PEF, making a small angle $d\theta$ with each other. Let $\angle \mathsf{CPD} = \theta$. With P as centre, and any radius $r > \mathsf{PB}$, describe an arc cutting PC and PF. Also, with radius $r + dr$, describe another arc. A small figure, approximately rectangular, is thus described, whose area $= rd\theta \times dr$.

We can now decompose the cylindrical conductor into an infinite number of linear elements of which the sections are similar to the above, and finding the field at P due to one of these, determine the total field at P by integration.

Let c be the current density in the conductor—that is, the current flowing through unit area of the conductor. Then current flowing through the above element $= crd\theta dr$.

This current will produce a field at P of magnitude $=\frac{2crd\theta dr}{r}$ (see page 171), in a direction Pf' perpendicular to PB—that is, making an angle θ with PQ, which is perpendicular to PAD, a line passing through P and the centre H of the circle ABC. There will be another element on the opposite side of AD, symmetrically situated with respect to PAD, which will produce an equal field at P in the direction Pf'', making an angle $-\theta$ with PQ. Consequently the components of Pf', Pf'', perpendicular to PQ, will annul each other, and we need only take account of the components parallel to PQ—that is, Pf' $\cos\theta = 2cd\theta dr \times \cos\theta$.

$\therefore$ Total field at P will be perpendicular to PAD, and of magnitude $2c\int\int\cos\theta d\theta dr$, the integration being performed over the surface bounded by the circle ADC.

To find the limits of r (that is, PB and PC) we have

$$\text{CH}^2 = \text{PH}^2 + \text{PC}^2 - 2\text{PH}\cdot\text{PC}\cos\theta. \quad \text{(See Appendix.)}$$

Let CH $= a =$ radius of circle ABC.

Let PH $= b =$ distance of point P from the axis of the cylin drical conductor.

Let PC $= r$.

Then $a^2 = b^2 + r^2 - 2br\cos\theta$.

$$\therefore r^2 - 2br\cos\theta = a^2 - b^2.$$

Complete the square on the left-hand side of this equation by adding $b^2\cos^2\theta$, the same quantity being also added to the right-hand side of the equation. Then

$$\begin{aligned} r^2 - 2br\cos\theta + b^2\cos^2\theta &= a^2 - b^2 + b^2\cos^2\theta \\ &= a^2 - b^2(1-\cos^2\theta) \\ &= a^2 - b^2\sin^2\theta. \end{aligned}$$

Taking the square root of both sides of the equation, we get

$$r - b\cos\theta = \pm\sqrt{a^2 - b^2\sin^2\theta}.$$

$$\therefore r = b\cos\theta \pm \sqrt{a^2 - b^2\sin^2\theta}.$$

The lower sign will give the value of PB, the upper sign that of PC.

Integrating with respect to r between these limits will give the component magnetic field at P, perpendicular to PA, due to the current flowing along that part of the cylindrical conductor of which the section is EFCB. It will then remain to sum up the component fields due to the various elements, similar to EFCB, into which the cylindrical conductor may be decomposed.

It is obvious that θ must vary from the negative angle RPH (Fig. 57) to the positive angle SPH, where PR and PS are

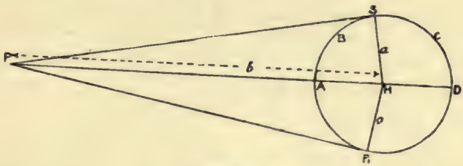

FIG. 57.

tangents to the circle ABCDR. Let $\theta_1 = \angle$SPH. Then, since $\angle \mathrm{HSP} = \frac{\pi}{2}$, we have

$$\sin\theta_1 = \frac{\mathrm{SH}}{\mathrm{PH}} = \frac{a}{b}.$$

$$\therefore\ \theta_1 = \sin^{-1}\frac{a}{b}.$$

Finally, the resultant magnetic field at P will be perpendicular to PH, and its magnitude will be given by the definite double integral

$$2c\int_{-\sin^{-1}\frac{a}{b}}^{\sin^{-1}\frac{a}{b}}\int_{b\cos\theta-\sqrt{a^2-b^2\sin^2\theta}}^{b\cos\theta+\sqrt{a^2-b^2\sin^2\theta}}\cos\theta\, d\theta dr.$$

Integrating with respect to r, and substituting the limits, we get

$$2c\int_{-\sin^{-1}\frac{a}{b}}^{\sin^{-1}\frac{a}{b}} 2(a^2-b^2\sin^2\theta)^{\frac{1}{2}}\cos\theta\, d\theta.$$

To effect this latter integration, let $b\sin\theta = x$.

$$\therefore\ \cos\theta\, d\theta = \frac{dx}{b}.$$

The limits of x are given by the relation

$$b \sin \theta = x_1 = a, \text{ and}$$
$$x_2 = -a.$$

Substituting in the integral to be evaluated, we obtain

$$\frac{4c}{b}\int_{-a}^{+a}(a^2-x^2)^{\frac{1}{2}}dx = \frac{4c}{b}\Big[_{-a}^{+a}\,(a^2-x^2)^{\frac{1}{2}}\frac{x}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\Big]$$
$$=\frac{4c}{b}\cdot a^2\frac{\pi}{2}=\pi a^2 c\,\frac{2}{b}.$$

Let $\pi a^2 c$, which is equal to the total current flowing through the conductor, be equal to C. Then the magnetic field at a distance b from the centre of the conductor is equal to

$$\frac{2\text{C}}{b}.$$

This is equal to the magnetic field, calculated on the assumption that the whole current flows through a thin conducting filament coincident with the axis of the given conductor.

By reasoning similar to that used above, the field at P (Fig. 58),

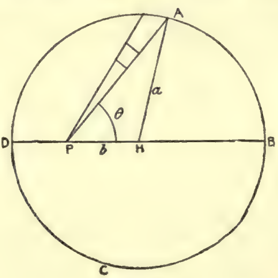

FIG. 58.

a point inside the conductor, is perpendicular to line PH, passing through centre of conductor, and its magnitude is given by

$$2c\iint \cos\theta\, d\theta dr,$$

the integration extending over the circular area ABCD.

Let $a =$ radius of circle,
$b =$ distance PH,
$r =$ PA.
Then $a^2 = r^2 + b^2 - 2rb \cos \theta.$
Hence $r = b \cos \theta \pm \sqrt{a^2 - b^2 \sin^2 \theta}.$

The upper sign obviously applies to the present case. Hence the required magnetic field

$$= 2c \int_{-\pi}^{+\pi} \int_0^{b \cos \theta + \sqrt{a^2 - b^2 \sin^2 \theta}} \cos \theta \, d\theta dr$$

$$= 2c \int_{-\pi}^{+\pi} (b \cos \theta + \sqrt{a^2 - b^2 \sin^2 \theta}) \cos \theta \, d\theta$$

$$= 2c \int_{-\pi}^{+\pi} b \cos^2 \theta \, d\theta,$$

the quantity $\sqrt{a^2 - b^2 \sin^2 \theta} \cdot \cos \theta \, d\theta$ being equal to zero when integrated between the limits $-\pi$ and $+\pi$. (See above, p. 182.)

$$2c \int_{-\pi}^{+\pi} b \cos^2 \theta \, d\theta = 2cb \int_{-\pi}^{+\pi} \frac{1 + \cos 2\theta}{2} d\theta = 2\pi cb.$$

Also, $c = \dfrac{\text{C}}{\pi a^2}.$

$\therefore$ Magnetic field at $\text{P} = \dfrac{2\pi b \times \text{C}}{\pi a^2} = \dfrac{2b\text{C}}{a^2}.$

It will be noticed that as the magnetic field in each of the above cases depends for its value only on the distance from the axis of the conductor, the lines of force will be circles.

Triple Integration.

As already explained, the position of a point in a plane becomes known when its distances from two straight lines at right angles to each other are given. In the same manner, the position of a point in *space* becomes known when its distances from three fixed planes, each one of which is perpendicular to the other two, are given.

Let OX, OY, OZ (Fig. 59) be the lines of intersection of three planes, each one of which is perpendicular to the other two. Let P be any point in space. Then the position of P is known when the lengths of the lines AP, BP, and CP, drawn perpendicular to the planes YOX, XOZ, and ZOY respectively, are given.

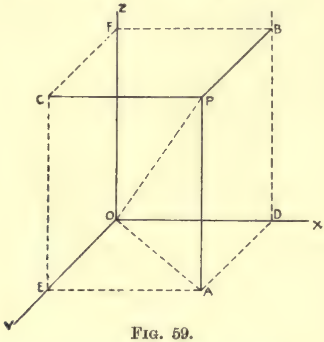

FIG. 59.

Complete the rectangular parallelopiped PO. Then $CP = OD$, $BP = OE$, and $AP = OF$. If we define the straight lines OX, OY, and OZ, forming the lines of intersection of the three reference planes, as the axes of x, y, and z respectively, we can write $CP = OD = x$, $BP = OE = y$, $AP = OZ = z$. Then the position of the point P becomes definitely known when the corresponding values of x, y, and z are given. The distances x, y, and z are termed the rectangular or Cartesian co-ordinates of the point P.

The distance from the origin of the point P can be easily expressed in terms of its rectangular co-ordinates x, y, z. Join O and A by a straight line. Then, since PA is perpendicular to the plane YOX, it is perpendicular to the line OA drawn in that plane. Hence PAO is a right-angled triangle.

$$\therefore\ OP^2 = OA^2 + AP^2 = OA^2 + OF^2.$$

But OAD is a right-angled triangle, the right angle being at D. Therefore

$$OA^2 = OD^2 + DA^2 = OD^2 + OE^2.$$

$$\therefore\ OP^2 = OD^2 + OE^2 + OF^2.$$

Let r be the distance OP of the point P from the origin O. Then
$$r^2 = x^2 + y^2 + z^2.$$

If any surface be described in space subject to the condition that all points on it are at a constant distance from the origin O, then the surface described is termed a sphere. If x, y, z are the co-ordinates of any point on a sphere with centre at the origin, then the relation between x, y, z is given in the equation

$$x^2+y^2+z^2=\text{constant}=a^2 \text{ (say)}.$$
$$\therefore\ x^2+y^2+z^2-a^2=0.$$

This equation may be termed the *equation to the sphere* of which a is the radius, the centre being at the origin.

If any surface is described in space, then the equation connecting the co-ordinates x, y, z of any point on the surface is termed the equation to the surface. In other words, the equation to any surface may be expressed in the general form

$$\mathsf{F}(x, y, z)=0.$$

The meaning and method of dealing with a triple integral will now be illustrated during the solution of a few problems.

Let $\mathsf{F}(x, y, z)$ be the equation to a surface. It is required to determine the volume bounded by part of the surface and the three reference planes YOX, XOZ, ZOY.

Let GK, KL, LG (Fig. 60) be sections of the surface by the

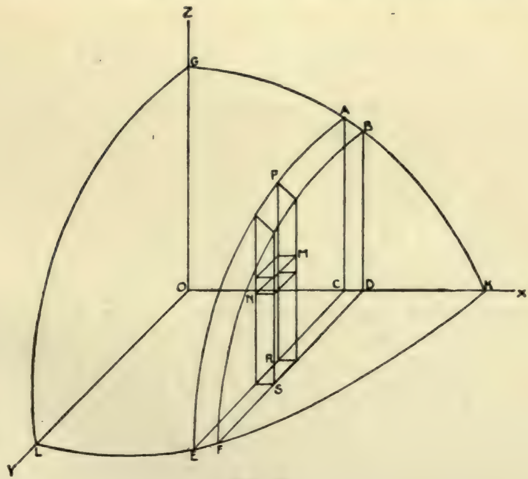

FIG. 60.

three reference planes, and let P be any point (x, y, z) on the surface. Draw a plane APEC parallel to the plane YOZ, through the point P; then $OC = x$. Take a point D at a distance from C, along the axis of x, equal to dx, and draw the plane BDF parallel to APEC through this point. Then the two planes APEC and BDF cut off a slice, of thickness dx, from the solid. This slice may be subdivided into columns by planes parallel to the plane ZOX, and at distances equal to dy apart; PS is one of these columns. Each column may be further subdivided into rectangular parallelopipeda by means of planes parallel to the plane YOX, and at distances equal to dz apart. MN is one such parallelopipedon, of which the edges, parallel to the axes of x, y, and z, are respectively equal to dx, dy, dz. Then the volume of MN is equal to $dxdydz$.

The value of $z = RP$ can be found, in terms x and y, from the equation to the surface, $F(x, y, z) = 0$. Let $z = f(x, y)$. Then, to determine the volume of the column PS, we must integrate $dxdydz$, with respect to z, between the limits $z = 0$ and $z = RP = f(x, y)$, treating dy and dx as constants, since the sections of the column, by planes parallel to YOX, are uniform.

$$\therefore \text{ Volume of column PS} = \int_0^{f(x, y)} dxdydz.$$

As already explained, the slice AF may be considered to consist of columns similar to PS. If we sum up the volumes of these constituent columns, we obtain the volume of the slice AF. To do this, we must integrate the expression for the volume of the column PS, with respect to y, between the limits $y = 0$ and $y = CE$. The value of $y = CE$ may be determined in terms of x, if we notice that all points on the curve LEK lie on the surface $F(x, y, z)$, but that for each point on that curve $z = 0$. Hence, if we substitute 0 for z in the equation $F(x, y, z) = 0$, we obtain a relation between x and y, from which y may be determined in terms of x. Let $\phi(x)$ be the value of y so determined. Then we must integrate the expression for the volume of a column such as PS, with respect to y, between the limits $y = 0$ and $y = \phi(x)$.

$$\therefore \text{ Volume of slice AF} = \int_0^{\phi(x)} \int_0^{f(x, y)} dxdydz.$$

Now, the volume GKLO may be considered to consist of an infinite number of slices such as AF. We have just found an expression for the volume of one such slice, and the volume of any particular slice can be found by substituting the corresponding value OC of x. To find the total volume of GKLO, we must integrate the expression for the volume of a slice, with respect to x, between the limits $x=0$ and $x=\text{OK}=k$ (say).

$$\therefore \text{ Volume of solid GKLO} = \int_0^k \int_0^{\phi(x)} \int_0^{f(x,\,y)} dxdydz.$$

Application to the Case of a Sphere.

Let $x^2+y^2+z^2=a^2$ be the equation to the sphere.

Then, since $f(x, y)$ is the value of z, expressed in terms of x and y, determined from the equation to the surface, we have

$$z=f(x,\,y)=(a^2-x^2-y^2)^{\frac{1}{2}}.$$

$\phi(x)$ is equal to the value of y, determined from the same equation, when 0 is substituted for z.

$$\therefore\; y=\phi(x)=(a^2-x^2)^{\frac{1}{2}}.$$

Finally, OK will be equal to the radius of the sphere—that is, to a. Hence, the volume of one-eighth part of a sphere

$$=\int_0^a \int_0^{(a^2-x^2)^{\frac{1}{2}}} \int_0^{(a^2-x^2-y^2)^{\frac{1}{2}}} dxdydz.$$

Integrating this expression with respect to z, we get

$$\int_0^a \int_0^{(a^2-x^2)^{\frac{1}{2}}} \left\{ \Big[z \Big]_0^{(a^2-x^2-y^2)^{\frac{1}{2}}} \right\} dxdy$$

$$=\int_0^a \int_0^{(a^2-x^2)^{\frac{1}{2}}} (a^2-x^2-y^2)^{\frac{1}{2}} dxdy.$$

Integrating with respect to y, we must treat x and dx as constants. The result obtained will be equal to

$$\int_0^a \left\{ \left[\frac{y}{2}(a^2-x^2-y^2)^{\frac{1}{2}} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}} \right]_0^{(a^2-x^2)^{\frac{1}{2}}} \right\} dx.$$

(See page 119.)

Substituting the limiting values, we find that the first term is equal to 0 when $y=0$, and also when $y=(a^2-x^2)^{\frac{1}{2}}$. When $y=0$ is substituted in the second term, we get $\sin^{-1}0=0$, since the sine of zero angle is equal to zero. When the upper limit is substituted for y, the second term becomes equal to

$$\frac{(a^2-x^2)}{2}\sin^{-1}\frac{(a^2-x^2)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{1}{2}}}=\frac{a^2-x^2}{2}\sin^{-1}1=\frac{a^2-x^2}{2}\times\frac{\pi}{2},$$

$$\text{since } \sin\frac{\pi}{2}=1.$$

Therefore, after integrating with respect to y, and substituting the limits, we get for the volume of one-eighth of a sphere the expression

$$\frac{\pi}{4}\int_0^a(a^2-x^2)dx=\frac{\pi}{4}\Big[a^2x-\frac{x^3}{3}\Big]_0^a=\frac{\pi}{4}\left(a^3-\frac{a^3}{3}\right)=\frac{\pi a^3}{6}.$$

The volume of the complete sphere will therefore be equal to

$$8\times\frac{\pi a^3}{6}=\frac{4}{3}\pi a^3,$$

a result we have already obtained, using another method. (See page 106.)

Application to the Case of an Ellipsoid.

The surface represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is termed an ellipsoid. It can easily be proved that the section by any reference plane will be an ellipse. Take, for instance, the section by the plane ZOX. For all points in this plane $y=0$. Substituting this value of y in the equation to the surface, we get, for the equation to the section by the plane ZOX,

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1.$$

This is the equation to an ellipse, of which the principal semi-axes are equal to a and c, the origin being the centre. (See page 20.) In a similar manner, it can be proved that the sections by the other reference planes are ellipses.

The volume bounded by the ellipsoid and the three reference planes may be determined in a manner similar to that used in the case of the sphere.

$$f(x, y) = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} = \frac{c}{b}\left(b^2 - \frac{b^2x^2}{a^2} - y^2\right)^{\frac{1}{2}}$$

$$\phi(x) = b\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{2}} = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}.$$

$$k = \text{OK (Fig. 60)} = a.$$

This last result follows from the fact that when the values $y=0$, $z=0$ are substituted in the equation to the ellipsoid, we get

$$\frac{x^2}{a^2} = 1. \quad \therefore x = \pm a.$$

The positive sign will obviously correspond to the positive value OK of x.

The integrations to be performed may be left as an exercise to the student. The method to be pursued is similar, step by step, to that fully worked out in the preceding problem. The result obtained is, that the volume bounded by the ellipsoidal surface and the three reference planes is equal to

$$\frac{\pi abc}{6}.$$

Since the surface is symmetrical about the centre, the total volume bounded by the ellipsoid will be equal to

$$\frac{4}{3}\pi abc.$$

A sphere may be considered to be a particular ellipsoid, in which the three principal semi-axes are equal in magnitude.

Problem.—Determine the moment of inertia of a solid ellipsoid of uniform density, about one of the principal axes.

Let GKLO (Fig. 60) represent one of the eight equal parts into which the ellipsoid may be divided by the three reference planes, OK, OG, OL being the principal axes of the ellipsoid, coinciding with the axes of x, y, and z respectively. Let it be required to determine the moment of inertia of the ellipsoid about the axis of z.

We must divide the solid into an indefinitely large number of elements, find the mass of each element, and then determine the sum of the products of these elements of mass into the squares of their respective distances from the axis of z.

Let $\rho =$ the density of the substance of which the ellipsoid is composed. Then, if we divide the ellipsoid up in the manner described on page 186, the volume of the element MN (Fig. 60) will be $dxdydz$, and its mass will be equal to $\rho dxdydz$. Also, all points on the line PR will be at a perpendicular distance $y = \mathrm{CR}$ from the plane ZOX, and at a perpendicular distance $x = \mathrm{OC}$ from the plane YOZ. The perpendicular distance of any point on the line PR from the axis OZ will obviously be equal to $\sqrt{\mathrm{OC}^2 + \mathrm{CR}^2} = \sqrt{x^2 + y^2}$. Hence we may take the perpendicular distance of the small element of volume MN from the axis of z as equal to $\sqrt{x^2 + y^2}$. Thus the element of the moment of inertia of the solid, due to the element of volume MN, is equal to $\rho dxdydz \times (x^2 + y^2)$.

We must integrate this result throughout the whole of the volume of the ellipsoid, thus summing up the elements of the moment of inertia due to the various elements of volume into which the ellipsoid has been decomposed. It can easily be seen that the three reference planes divide the ellipsoid into eight equal parts, and that in each of these parts there will be an element of volume similar to MN, and possessing co-ordinates of the same *numerical* value as MN, the only differences being the signs affixed to x, y, and z. The distance of all of these elements from the axis of z will be equal to $(x^2 + y^2)$. Hence we may find the moment of inertia of the whole ellipsoid by integrating $\rho(x^2 + y^2)dxdydz$ throughout the volume GKLO (Fig. 60), and then multiplying the result obtained by 8.

The limits of the integration will be the same as those used in finding the volume of the ellipsoid. Hence the moment of inertia of the ellipsoid will be equal to

$$8\rho \int_0^a \int_0^{\frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}} \int_0^{\frac{c}{b}\left(b^2 - \frac{b^2x^2}{a^2} - y^2\right)^{\frac{1}{2}}} (x^2 + y^2)dxdydz.$$

Integrating this with respect to z, and treating x, y, dx, dy as constants, we get

$$\frac{8\rho c}{b}\int_0^a\int_0^{\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}}(x^2+y^2)\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dxdy.$$

In order to integrate with respect to y, we must determine the value of the two following indefinite integrals—

$$x^2\int\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy+\int y^2\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy.$$

Remembering that we must treat x as a constant during the integration with respect to y, we have

$$x^2\int\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy$$

$$=x^2\left\{\frac{y}{2}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}+\frac{b^2-\dfrac{b^2x^2}{a^2}}{2}\sin^{-1}\frac{y}{\sqrt{b^2-\dfrac{b^2x^2}{a^2}}}\right\}\quad\ldots\quad(a)$$

Also, on integrating by parts, we have

$$\int y^2\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy=\int y\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}ydy$$

$$=-\frac{y}{3}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}+\frac{1}{3}\int\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}dy$$

$$=-\frac{y}{3}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}+\frac{1}{3}\int\left\{\left(b^2-\frac{b^2x^2}{a^2}\right)-y^2\right\}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy$$

$$=-\frac{y}{3}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}+\frac{1}{3}\left(b^2-\frac{b^2x^2}{a^2}\right)\int\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy$$

$$-\frac{1}{3}\int y^2\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy.$$

The value of the first integral in this expression has been written down above. Taking the second integral over to the left-hand side of the equation, we have

$$\frac{4}{3}\int y^2\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy = -\frac{y}{3}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}$$

$$+\frac{1}{3}\left(b^2-\frac{b^2x^2}{a^2}\right)\cdot\left\{\frac{y}{2}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}+\frac{b^2-\frac{b^2x^2}{a^2}}{2}\sin^{-1}\frac{y}{\sqrt{b^2-\frac{b^2x^2}{a^2}}}\right\}.$$

$$\therefore \int y^2\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dy = \frac{y}{4}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{3}{2}}$$

$$+\frac{1}{4}\left(b^2-\frac{b^2x^2}{a^2}\right)\left\{\frac{y}{2}\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}+\frac{b^2-\frac{b^2x^2}{a^2}}{2}\sin^{-1}\frac{y}{\sqrt{b^2-\frac{b^2x^2}{a^2}}}\right\}(b)$$

Substituting the limiting values of y—namely,

$$y=\frac{b}{a}(a^2-x^2)^{\frac{1}{2}} \text{ and } y=0,$$

we find that (b) reduces to

$$\frac{\left(b^2-\frac{b^2x^2}{a^2}\right)^2}{8}\sin^{-1}1=\frac{\left(b^2-\frac{b^2x^2}{a^2}\right)^2}{8}\cdot\frac{\pi}{2}.$$

Similarly (a) above reduces to

$$x^2\frac{b^2-\frac{b^2x^2}{a^2}}{2}\cdot\frac{\pi}{2}.$$

$$\therefore \frac{8\rho c}{b}\int_0^a\int_0^{\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}}(x^2+y^2)\left(b^2-\frac{b^2x^2}{a^2}-y^2\right)^{\frac{1}{2}}dxdy$$

$$=\frac{8\rho c}{b}\int_0^a\left\{\frac{\pi}{4}\cdot x^2\left(b^2-\frac{b^2x^2}{a^2}\right)+\frac{\pi}{16}\left(b^2-\frac{b^2x^2}{a^2}\right)^2\right\}dx$$

$$=\frac{2\rho\pi c}{b}\int_0^a\left\{\frac{b^4}{4}+\left(b^2-\frac{b^4}{2a^2}\right)x^2+\left(\frac{b^4}{4a^4}-\frac{b^2}{a^2}\right)x^4\right\}dx$$

$$=\frac{2\rho\pi c}{b}\left\{\frac{b^4a}{4}+\left(b^2-\frac{b^4}{2a^2}\right)\frac{a^3}{3}+\left(\frac{b^4}{4a^4}-\frac{b^2}{a^2}\right)\frac{a^5}{5}\right\}$$

$$=\frac{2\rho\pi c}{b}\left(\frac{2b^2a^3}{15}+\frac{2b^4a}{15}\right)=\frac{4\pi abc}{3}\rho\left(\frac{a^2+b^2}{5}\right)$$

$$=\mathsf{M}\left(\frac{a^2+b^2}{5}\right),\text{ if }\mathsf{M}=\frac{4}{3}\pi abc\rho=\text{the mass of the ellipsoid.}$$

Problem.—Determine the moment of inertia of a thin uniform ellipsoidal shell, the semi-axes of which are equal to a, b, and c, about a principal axis.

Let t be the uniform thickness of the shell, and let ρ be the density of the substance composing the shell.

Then, mass of a solid ellipsoid, of which the semi-axes are equal to a, b, and c,

$$=\frac{4}{3}\pi\rho abc.$$

Mass of a solid ellipsoid, of which the semi-axes are equal to $(a+t)$, $(b+t)$, $(c+t)$,

$$=\frac{4}{3}\pi\rho(a+t)(b+t)(c+t)$$

$$=\frac{4}{3}\pi\rho\left(abc+(ab+bc+ac)t+(a+b+c)t^2+t^3\right)$$

$$=\frac{4}{3}\pi\rho\left(abc+(ab+bc+ac)t\right),$$

if t is so small that terms involving t^2 and t^3 may be neglected.

$\therefore$ Mass of ellipsoidal shell of thickness t

$$=\frac{4}{3}\pi\rho(ab+bc+ac).$$

Also, moment of inertia of solid ellipsoid, of which the semi-axes are equal to a, b, c, about the axis c,

$$=\frac{4\pi}{15}\rho(a^3bc+b^3ac).$$

Moment of inertia of solid ellipsoid, of which the semi-axes are equal to $(a+t)$, $(b+t)$, $(c+t)$,

$$=\frac{4\pi}{15}\rho\{(a+t)^3(b+t)(c+t)+(b+t)^3(a+t)(c+t)\}.$$

If t is so small that terms involving t^2 and higher powers of t may be neglected, we may write

$$(a+t)^3=a^3+3a^2t,$$
$$(b+t)^3=b^3+3b^2t.$$

Hence moment of inertia of ellipsoid, of which the semi-axes are equal to $(a+t)$, $(b+t)$, $(c+t)$,

$$=\frac{4\pi\rho}{15}\{(a^3+3a^2t)(b+t)(c+t)+(b^3+3b^2t)(a+t)(c+t)\}$$
$$=\frac{4\pi\rho}{15}\{a^3bc+b^3ac+(a^3b+a^3c+3bca^2+b^3a+b^3c+3acb^2)t+\ldots\},$$

neglecting terms involving t^2 and t^3.

$\therefore$ Moment of inertia of shell of thickness t

$$=\frac{4\pi\rho}{15}t\{a^3(b+c)+b^3(a+c)+3abc(a+b)\}.$$

This result does not admit of expression in terms of the mass of the ellipsoidal shell.

Exercises.—(1.) Determine the moment of inertia of an elliptical disc, of mass M, about an axis through the centre and at right angles to the disc.

Answer.—Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the equation to the ellipse. The required moment of inertia about the axis of y

$$=4\rho\int_0^a\int_0^{\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}}(x^2+y^2)dxdy=\mathrm{M}\,\frac{a^2+b^2}{4}.$$

(2.) Determine the moment of inertia of an elliptical disc, of mass M, about the minor axis.

Answer—

$$4\rho\int_0^a\int_0^{\frac{b}{a}(a^2-x^2)^{\frac{1}{2}}} x^2 dx dy = \mathrm{M}\,\frac{a^2}{4}.$$

(3.) Determine the moment of inertia of a cylinder of elliptical section, about an axis through its centre, at right angles to its length, and parallel to the minor axis of the elliptical section, the length of the cylinder being l, and the semi-axes of the section being b (major) and c (minor).

Answer—

$$8\rho\int_0^{\frac{l}{2}}\int_0^b\int_0^{\frac{c}{b}(b^2-y^2)^{\frac{1}{2}}} (x^2+y^2) dx dy dz = \mathrm{M}\left(\frac{l^2}{12}+\frac{b^2}{4}\right).$$

(4.) Solve Exercise No. 3 by the aid of the principle explained on page 157.

(5.) Determine the moment of inertia of a rectangular rod (the sides of which are equal to $2a$, $2b$, $2c$), about an axis through its centre, and perpendicular to the plane containing the sides of length $2b$ and $2c$.

Answer—

$$8\rho\int_0^c\int_0^b\int_0^a (x^2+y^2) dx dy dz = \mathrm{M}\cdot\frac{b^2+c^2}{3}.$$

(6.) Solve Exercise No. 5 by the aid of the principle explained on page 157.

(7.) Show that the co-ordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the centre of gravity of a body are given by

$$\bar{x} = \frac{\iiint x\, dx\, dy\, dz}{\iiint dx\, dy\, dz},$$

$$\bar{y} = \frac{\iiint y\,dx\,dy\,dz}{\iiint dx\,dy\,dz},$$

$$\bar{z} = \frac{\iiint z\,dx\,dy\,dz}{\iiint dx\,dy\,dz}.$$

Determine, by the aid of the formula obtained, the position of the centre of gravity of a solid hemisphere.

CHAPTER XI.

DIFFERENTIAL EQUATIONS.

AS already explained, the equation $y = a + bx$ represents a straight line inclined to the axis of x at an angle $\tan^{-1} b$, and passing through the point $y = a$, $x = 0$. In this equation b possesses a definite value, so that a certain definite straight line is represented. It is possible, however, to write down an equation which represents *any* straight line passing through a. In this case it is imperative, of course, that b should not occur.

We have

$$\frac{dy}{dx} = b.$$

Hence $y = a + \frac{dy}{dx} \cdot x$ is the required equation. Since it involves not only y and x, but also $\frac{dy}{dx}$, it is termed a *differential equation.*

In the above process we have eliminated b, and, in place of an equation involving y, a, b, and x, we have obtained an equation involving y, a, $\frac{dy}{dx}$, and x. The converse of this process is often necessary. Being given an equation involving y, x, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, etc., we are required to obtain an equation involving only y, x, and constants. The latter equation is termed the solution of the differential equation.

To solve the differential equation

$$y = a + \frac{dy}{dx} \cdot x,$$

we may proceed as follows :—

$$y - a = \frac{dy}{dx} \cdot x.$$

$$\therefore \frac{dy}{y-a} = \frac{dx}{x}.$$

We must now integrate both sides of this equation,

$$\int \frac{dy}{y-a} = \int \frac{dx}{x}.$$

$$\therefore \log_e(y-a) = \log_e x + \mathrm{C}.$$

We introduce the constant C on the right-hand side of this equation, so as to obtain the most general solution possible. It can easily be seen that the result obtained by differentiating both sides will be the same whether C is included or not. For this reason the most general value of an indefinite integral should always include an added arbitrary constant. It is unnecessary to add a constant to each side of the above equation, since, if we did so, the constant added on the left-hand side could be transferred to the right-hand side, and a new arbitrary constant substituted for the difference of the two constants which would then be included in the equation.

Let $$\mathrm{C} = \log_e b.$$

Then $$\log_e(y-a) = \log_e x + \log_e b = \log_e(bx).$$

$$\therefore y - a = bx,$$

$$y = a + bx.$$

Thus, starting with a differential equation, we have finally obtained an equation involving only y, x, and the constants a and b. In other words, we have solved the differential equation.

The equation

$$\frac{dy}{dx} = b$$

is itself a differential equation. It represents *any* straight line inclined at an angle, of which the tangent is b, to the axis of x. We may solve this equation as follows :—

$$dy = bdx,$$

$$\int dy = b \int dx.$$

$$\therefore y = bx + a.$$

Here the constant a has been added for the reasons explained above.

The way in which differential equations occur in the course of physical investigations will now be illustrated by the aid of a few simple examples.

Problem.—A body is moving toward the earth at a certain instant with a velocity V. Required the position and velocity of the body at a time t seconds afterwards.

Before proceeding to solve the above problem it may be well to indicate the way in which the velocity and acceleration of a body may be represented by the aid of differential coefficients.

At a certain instant let a body pass through the point O (Fig. 61) in the direction OA. After t seconds let the body pass

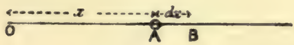

FIG. 61.

through A, at a distance x from O; and after a further indefinitely short interval of time, dt, let the body pass through B, at a distance $x + dx$ from O. Then, in the time dt, the body has traversed the distance dx; and since dt is an excessively small element of time, we may assume the velocity of the body to have remained practically constant between A and B, whether it was moving uniformly or not.

Hence the velocity of the body between A and B was equal to $\frac{dx}{dt}$. Expressed slightly differently, we may say that at the time t the velocity of the body was equal to $\frac{dx}{dt}$.

Let v be the velocity of a body at a time t. Then, if the velocity has increased to $v + dv$, after an indefinitely short interval of time, dt, the rate of increase in the velocity of the body, or, in other words, its acceleration, was equal to $\frac{dv}{dt}$ at the time t.

But we have found above that for v we may write $\frac{dx}{dt}$. Hence, at the time t, the acceleration of the body was equal to

$$\frac{d\frac{dx}{dt}}{dt} = \frac{d^2x}{dt^2}.$$

It must be carefully noted that $\frac{dx}{dt}$ represents the velocity *in the + direction of* x; similarly, $\frac{d^2x}{dt^2}$ represents the acceleration in the same direction.

To return to our problem. Take the initial position of the body as origin, and measure x vertically downwards. Then, since the force which gravity exerts on a body is constant as long as its distance from the centre of the earth is not greatly altered, the acceleration or rate of change of velocity of the body remains constant. Let g be the acceleration due to gravity. (In England g is equal to 981 cm./sec^2, or 32 ft./sec^2. Our solution will be given in centimetres or feet, according as one or the other of these values is used for g.)

$$\therefore \frac{d^2x}{dt^2} = g.$$

This is the differential equation to the motion of the body. Multiply both sides of this equation by dt, and integrate.

$$\int \frac{d^2x}{dt^2}\,dt = g\int dt.$$

$$\therefore \frac{dx}{dt} = gt + a.$$

To find the value of the constant a, notice that the initial value of $\frac{dx}{dt}$, or the velocity of the body toward the earth, was V. Thus, when $t=0$, $\frac{dx}{dt} = \mathsf{V}$.

$$\therefore \mathsf{V} = g \times 0 + a.$$

$$\therefore a = \mathsf{V}.$$

Hence, at any time t afterwards, we have

$$\frac{dx}{dt} = \mathsf{V} + gt.$$

$$\therefore dx = \mathsf{V}\,dt + gt\,dt.$$

$$\therefore \int dx = \mathsf{V}\int dt + g\int t\,dt.$$

$$\therefore x = \mathsf{V}t + g\frac{t^2}{2} + b.$$

To find the value of b, notice that when $t=0$, $x=0$.

$$0=V\times 0+g\times 0+b.$$
$$\therefore b=0.$$

Hence the solution of our problem is obtained in the form

$$x=Vt+\tfrac{1}{2}gt^2.$$

Corollary.—If the body was moving *away* from the earth with a velocity V at the instant $t=0$, we must write

$$\frac{dx}{dt}=-V \text{ when } t=0.$$

The solution is then obtained in the form

$$x=-Vt+\tfrac{1}{2}gt^2.$$

To find the greatest altitude reached by the body, note that when that altitude is reached the velocity of the body will, for an instant, be equal to 0.

$$0=gt-V. \quad \therefore t=\frac{V}{g}\cdot$$

At this time

$$x=-\frac{V^2}{g}+\tfrac{1}{2}g\left(\frac{V}{g}\right)^2=-\tfrac{1}{2}\frac{V^2}{g}\cdot$$

The meaning of the negative sign is, that the distance numerically equal to $\frac{1}{2}\frac{V^2}{g}$ must be measured *upwards* from the point corresponding to the time $t=0$.

Problem.—Determine the equation of a plane curve such that the radii of curvature at all points on it shall be equal.

It has been proved, on page 84, that if $y=f(x)$ is the equation to a curve, the radius of curvature r, at a point x, y on the curve, is given by the equation

$$r=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}\cdot$$

In the present case, since the radii of curvature at all points on the curve are to be equal, we must treat r as a constant. The equation to the curve of constant curvature will be the solution of the differential equation

$$\frac{\frac{d^2y}{dx^2}}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\frac{1}{r}.$$

Multiply both sides of this equation by $\frac{dy}{dx}\cdot dx$, and integrate.

$$\int\frac{\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot dx}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\frac{\int\frac{dy}{dx}\,dx}{r}.$$

If it be remarked that $\frac{d}{dx}\cdot\left(\frac{dy}{dx}\right)^2=2\,\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$, no difficulty will be experienced in obtaining the results of the integrations indicated in the following form :—

$$\frac{1}{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}=\frac{y-b}{r}.$$

On the right-hand side of this equation we have subtracted the constant $\frac{b}{r}$, instead of adding any other arbitrary constant. No generality is thus lost, since b may have any value, positive or negative.

Squaring both sides and transposing, we get

$$1+\left(\frac{dy}{dx}\right)^2=\frac{r^2}{(y-b)^2},$$

$$\frac{dy}{dx}=\pm\left\{\frac{r^2}{(y-b)^2}-1\right\}^{\frac{1}{2}}=\pm\frac{(r^2-(y-b)^2)^{\frac{1}{2}}}{y-b}.$$

$$\therefore\ \pm\frac{(y-b)dy}{\{r^2-(y-b)^2\}^{\frac{1}{2}}}=dx.$$

Integrating this equation, we get

$$\pm \int \frac{(y-b)dy}{\{r^2-(y-b)^2\}^{\frac{1}{2}}} = \int dx.$$

Using the lower $(-)$ sign on the left-hand side of this equation, we get

$$\{r^2-(y-b)^2\}^{\frac{1}{2}} = x - a,$$

where we have added the arbitrary constant $-a$ to the right-hand side of the equation. If we had used the upper $(+)$ sign prefixed to the left-hand integral above, we should have obtained

$$-\{r^2-(y-b)^2\}^{\frac{1}{2}} = x - a.$$

Squaring both sides, we obtain, in either case,

$$r^2-(y-b)^2=(x-a)^2.$$
$$\therefore\ (x-a)^2+(y-b)^2=r^2.$$

This is the equation of a circle (see page 16), the centre being at the arbitrary point $(x=a,\ y=b)$, and the radius being equal to r.

It should be noticed that since the differential equation to be solved involved the second differential coefficient of y—that is, $\frac{d^2y}{dx^2}$—two integrations were necessarily performed during its solution, and two arbitrary constants $(-a$ and $-b)$ were thus introduced into the solution. The minus signs were chosen for the constants, in order to throw the final result into a form similar to that obtained directly on page 16. No generality was thus lost, since a or b, or both, may, in any particular case, be given negative value if necessary.

Problem.—The ends of a chain are fixed to two points in a horizontal straight line, at a definite distance apart. It is required to determine the equation to the curve in which the chain hangs.

The chain will obviously hang in a vertical plane. Let A, B (Fig. 62) be the two points in a horizontal straight line to which the ends of the chain are fixed, and let the chain hang in the curve ACB. Bisect AB in D, and draw DO perpendicular to AB. Take O as origin, and let OX, OX′ be the positive and negative directions of the axis of x, OY being the axis of y. Let $DB=x_1$, $OD=y_1$.

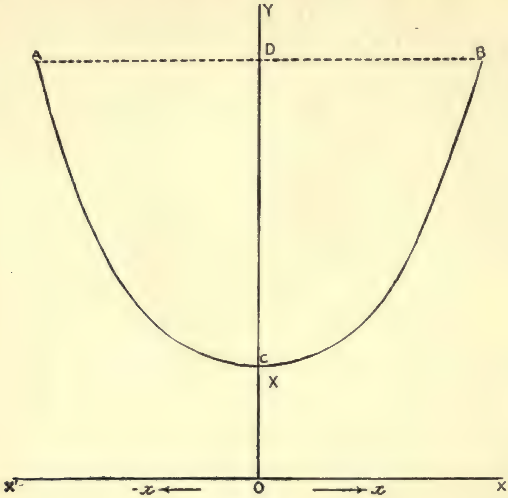

FIG. 62.

To obtain the differential equation to the curve ACB, we must write down the conditions that each element of the chain shall be in equilibrium under the action of the forces—(1) the weight of the element acting vertically downwards; (2) the tension of the chain at the upper extremity of the element; and (3) the tension of the chain at the lower extremity of the element.

Let AB (Fig. 63) be an element of the chain. Then, since this element is in equilibrium, we may imagine it to become rigid. If the length of AB be denoted by ds, and if w be the weight of unit length of the chain, then the pull of gravity on the element AB will act at the centre point of that element, and its magnitude will be equal to wds.

The tension of the chain at the point A will act along the tangent to the curve at that point. Let its magnitude be represented by T_1. Similarly, the tension of the chain at B may be represented by T_2, acting along the tangent to the curve at B. Then the element AB is in equilibrium under the action of the forces wds, T_1, and T_2. Hence these three forces must meet at a point C (Fig. 63).

The conditions of equilibrium of AB are—(1) the resultant of the vertical components of the three forces acting on AB must be equal to zero; (2) the resultant of the horizontal components of the three forces must be equal to zero. These conditions are necessary, since, if either resultant had any finite magnitude, the element AB would move in the direction of that resultant.

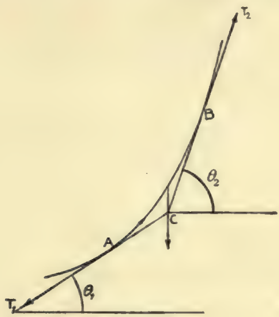

FIG. 63.

Let T_1 be inclined at an angle θ_1 to the axis OX, and let T_2 be inclined at an angle θ_2 to OX. The condition (1) gives

$$T_1 \sin \theta + wds = T_2 \sin \theta_2 \quad . \quad . \quad . \quad (1).$$

The condition (2) gives

$$T_1 \cos \theta_1 = T_2 \cos \theta_2 \quad . \quad . \quad (2),$$

since the pull of gravity on AB is vertical, and therefore has no horizontal component.

Dividing the left-hand side of (1) by $T_1 \cos \theta_1$, and the right-hand side by the equal quantity $T_2 \cos \theta_2$, we get

$$\tan \theta_1 + \frac{wds}{T_1 \cos \theta_1} = \tan \theta_2 \quad . \quad . \quad . \quad . \quad (3).$$

Equation (2) shows that the horizontal component of the tension is constant for all points on the chain. For we may divide the whole of the chain up into consecutive elements, and the horizontal components of the tension at the ends of each element will be equal, and the tension at the upper extremity of one element will obviously be equal to the tension at the lower extremity of the next higher element. Hence $\mathsf{T}_1 \cos \theta_1$ will have a constant value for all points on the chain. We may therefore write

$$\frac{w}{\mathsf{T}_1 \cos \theta_1} = \frac{1}{c},$$

and the equation (3) reduces to

$$\frac{ds}{c} = \tan \theta_2 - \tan \theta_1 . \quad . \quad . \quad (4).$$

Now $\tan \theta$ will obviously be a function of x, the abscissa of the corresponding point on the curve. Let the abscissa of A be x, and let the abscissa of B be $x + dx$, the element AB being now supposed to be indefinitely small. Then we may write

$$\tan \theta_2 - \tan \theta_1 = f(x+dx) - f(x) = f(x) + dxf'(x) + \ldots - f(x),$$

expanding $f(x+dx)$ by Taylor's theorem.

$$\therefore \tan \theta_2 - \tan \theta_1 = dxf'(x).$$

But $f(x) = \tan \theta_1$, and the tangent of the angle of inclination of the geometrical tangent at the point A on the curve is equal to $\frac{dy}{dx}$. (See page 28.) Hence $f'(x) = \frac{d^2y}{dx^2}$, and $\tan \theta_2 - \tan \theta_1 = \frac{d^2y}{dx^2} \cdot dx$.

Also, since (x, y) are the co-ordinates of A, and $(x + dx, y + dy)$ are the co-ordinates of B,

Length of arc AB

$$= ds = \{(dx)^2 + (dy)^2\}^{\frac{1}{2}} = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx.$$

Substituting in (4), we get

$$\frac{1}{c}\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx = \frac{d^2y}{dx^2} dx \ldots \quad . \quad . \quad (5).$$

Dividing through by dx, we get the differential equation to the curve in which the chain hangs in the form

$$\frac{d^2y}{dx^2} = \frac{1}{c}\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} . \quad . \quad (6).$$

In order to solve this equation we may proceed as follows :—

Dividing (5) through by $\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}$, multiplying both sides by $\frac{dy}{dx}$, and integrating, we get

$$\int \frac{\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}\cdot dx}{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{\frac{1}{2}}} = \frac{1}{c}\int\frac{dy}{dx}\,dx.$$

$$\therefore\ \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \frac{y}{c}+a,$$

where a is an arbitrary constant.

To find the value of a, notice that at the lowest point C (Fig. 62) of the curve ABC, $\frac{dy}{dx}=0$. Then the value of $y=\text{OC}$ is given by

$$\frac{y}{c}+a=1.$$

If we arrange that the origin O shall be at a distance equal to c, below the lowest point C of the chain, we have

$$\frac{c}{c}+a=1. \quad \therefore\ a=0.$$

Hence, subject to the condition just mentioned, the first integral of (6) may be written

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \frac{y}{c}.$$

We can now proceed to integrate this equation, or we can substitute in (6), obtaining

$$\frac{d^2y}{dx^2}=\frac{1}{c^2}y.$$

Either of these is the differential equation of the curve in which the chain hangs, provided the lowest point on the chain is at a distance C above the origin. The latter differential equation has been given because it is one of an important class that will be more fully discussed subsequently.

(*a*) To integrate

$$\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}=\frac{y}{c}.$$

Squaring and transforming, we get

$$\left(\frac{dy}{dx}\right)^2=\frac{y^2-c^2}{c^2},$$

$$\frac{dy}{dx}=\frac{\sqrt{y^2-c^2}}{c}.$$

$$\therefore \int\frac{dy}{(y^2-c^2)^{\frac{1}{2}}}=\int\frac{dx}{c}.$$

$$\therefore \log_\epsilon\{y+\sqrt{y^2-c^2}\}=\frac{x}{c}+b.$$

When $y=c$, and $x=0$,

$$\log_\epsilon(c)=b.$$

Therefore the above result may be written—

$$\log_\epsilon\{y+\sqrt{y^2-c^2}\}-\log c=\frac{x}{c}.$$

$$\therefore \log_\epsilon\frac{y+\sqrt{y^2-c^2}}{c}=\frac{x}{c}.$$

$$\therefore \frac{y+\sqrt{y^2-c^2}}{c}=\epsilon^{\frac{x}{c}}.$$

$$\therefore \frac{\sqrt{y^2-c^2}}{c}=\epsilon^{\frac{x}{c}}-\frac{y}{c}.$$

Squaring, we get

$$\frac{y^2-c^2}{c^2}=\epsilon^{\frac{2x}{c}}-\frac{2y\epsilon^{\frac{x}{c}}}{c}+\frac{y^2}{c^2}.$$

$$\therefore \frac{2y\epsilon^{\frac{x}{c}}}{c}=\epsilon^{\frac{2x}{c}}+1.$$

$$\therefore y=\frac{c}{2}\left(\epsilon^{\frac{x}{c}}+\epsilon^{-\frac{x}{c}}\right).$$

This is the equation to the curve in which the chain hangs. It is termed a *catenary*.*

(*b*) To integrate

$$\frac{d^2y}{dx^2}=\frac{1}{c^2}\,y.$$

Multiply both sides by $2\,\frac{dy}{dx}\,dx$, and integrate.

$$\int 2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2}\,dx=\frac{1}{c^2}\int y\,\frac{dy}{dx}\,dx.$$

$$\therefore \left(\frac{dy}{dx}\right)^2=\frac{y^2}{c^2}+a,$$

where a is an arbitrary constant.

At the point C (Fig. 62), where $y=c$, $\frac{dy}{dx}=0$.

$$\therefore\ 0=1+a.$$

$$\therefore\ a=-1.$$

$$\therefore\ \frac{dy}{dx}=\frac{\sqrt{y^2-c^2}}{c},$$

which agrees with the result of the first integration obtained above. Thus the final result is the same in whichever manner the integration is effected. This is a general rule. In certain circumstances, solutions which at first sight differ from one another may be obtained; but these solutions may always be transformed, the one into the other, by the use of suitable algebraical or trigonometrical devices.

Hyperbolic Sines, Cosines, etc.

In measuring an angle in radians, we may consider that the following procedure is followed. Given two lines intersecting each other, with their point of intersection as centre let an arc of a circle, cutting the two lines, be described. The portion of this circle between the two lines is then divided into an indefinitely large number of elements, the length of each element is obtained and divided by the radius of the circle, and, finally, the sum of all such terms is obtained. The value of this sum is termed the circular measure of the angle between the lines. If we denote

* Pronounced căte′nǎry. The derivation is from Lat. *catena*, a chain.

this by θ, and if s be the length of the arc measured between its points of intersection with the straight lines, ds the length of one of the infinitesimal circular arcs, and if r be the radius of the circle, we have

$$\theta = \int_0^s \frac{ds}{r}.$$

But since r is constant for all elements of the arc, we may write this result—

$$\theta = \frac{1}{r}\int_0^s ds = \frac{s}{r}.$$

The circle is the only plane curve of which all elements are at equal distances from any one point. In all other plane curves the distance of an element from any fixed point will vary with the position of the element. If, however, any point can be found which is symmetrically situated with regard to a curve, that point is termed the *centre* of the curve. Thus the point midway between the foci of an ellipse or hyperbola is termed the centre. It can easily be seen that if any straight line be drawn through this point, it will cut the curve in two points equidistant from, and on opposite sides of, the centre. On the other hand, a parabola has no centre.

The equation

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1, \text{ or } x^2 - y^2 = a^2,$$

represents a hyperbola, of which the origin is the centre, the two opposite branches of the curve intersecting the axis of x in the points $x = +a$ and $x = -a$ respectively. The distance from the centre to either focus is equal to $\sqrt{2} \cdot a$. (This can easily be proved in a manner similar to that used on page 20.)

Let us now divide the hyperbola up into an indefinitely great number of elements, find the length ds of each element, and divide this by its distance r from the centre. If we start from the point where the hyperbola cuts the axis of x, sum up the results thus obtained until a certain length s of the hyperbola has been passed over, and write ϕ for the sum so obta ned, we shall have

$$\phi = \int_0^s \frac{ds}{r}.$$

This result presents an obvious formal resemblance to the expression for the circular measure of an angle. ϕ does not, however, represent an angle; it is merely a certain function of the abscissa x of the terminal point of the hyperbolic arc. We shall now determine the relation between ϕ and x.

As proved on page 130,

$$ds = \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}} dx,$$

$$y = (x^2 - a^2)^{\frac{1}{2}},$$

$$\frac{dy}{dx} = \frac{x}{(x^2 - a^2)^{\frac{1}{2}}}.$$

$$\therefore\ 1 + \left(\frac{dy}{dx}\right)^2 = \frac{x^2 - a^2 + x^2}{x^2 - a^2} = \frac{2x^2 - a^2}{x^2 - a^2}.$$

$$\therefore\ ds = \frac{\sqrt{2x^2 - a^2}}{\sqrt{(x^2 - a^2)}}\, dx.$$

Also, r, the distance from the origin of the point (x, y) on the curve, is equal to $(x^2 + y^2)^{\frac{1}{2}}$.

$$r^2 = x^2 + y^2 = x^2 + x^2 - a^2 = 2x^2 - a^2.$$

$$\therefore\ r = \sqrt{2x^2 - a^2}.$$

$$\therefore\ \frac{ds}{r} = \frac{\sqrt{2x^2 - a^2}}{\sqrt{x^2 - a^2}}\, dx \div \sqrt{2x^2 - a^2} = \frac{dx}{\sqrt{x^2 - a^2}}.$$

Let x_1 be the abscissa of the ultimate point of this hyperbolic arc; as already pointed out, a will be the abscissa of the starting point. Thus,

$$\phi = \int_a^{x_1} \frac{dx}{(x^2 - a^2)^{\frac{1}{2}}} = {}_a^{x_1}\Big[\log_\epsilon(x + \sqrt{x^2 - a^2}) \Big]$$

$$= \log_\epsilon(x_1 + \sqrt{x_1^{\ 2} - a^2}) - \log_\epsilon a$$

$$= \log_\epsilon \frac{x_1 + \sqrt{x_1^{\ 2} - a^2}}{a}.$$

We may now write x for x_1, if we denote by x the abscissa of the terminal point of the hyperbolic arc. Hence,

$$\phi = \log_\epsilon \frac{x + \sqrt{x^2 - a^2}}{a}.$$

Therefore, from the definition of a logarithm,

$$\epsilon^\phi = \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a},$$

$$\frac{x^2 - a^2}{a^2} = \frac{x^2}{a^2} - 1 = \left(\epsilon^\phi - \frac{x}{a}\right)^2 = \epsilon^{2\phi} - \frac{2x}{a}\epsilon^\phi + \frac{x^2}{a^2}.$$

$$\therefore \frac{2x}{a}\epsilon^\phi = \epsilon^{2\phi} + 1.$$

$$\therefore \frac{x}{a} = \frac{\epsilon^\phi + \epsilon^{-\phi}}{2}.$$

Now, in accordance with the equation $x^2 + y^2 = r^2$, which represents a circle,

$$\frac{x}{r} = \cos\theta,$$

where θ is the angle which the radius drawn to the point of which the abscissa is x, makes with the axis of x. Also,

$$\frac{x}{r} = \cos\theta = \frac{\epsilon^{\iota\theta} + \epsilon^{-\iota\theta}}{2}.$$ (See page 80.)

An obvious similarity may be detected between this and the result found above—namely,

$$\frac{x}{a} = \frac{\epsilon^\phi + \epsilon^{-\phi}}{2},$$

where x is the abscissa of a point on the hyperbola

$$x^2 - y^2 = a^2.$$

Hence the quantity $\frac{x}{a}$ is termed the hyperbolic cosine, or the hypocosine of ϕ. For brevity we write—

$$\frac{\epsilon^\phi + \epsilon^{-\phi}}{2} = \cosh\phi.$$

For similar reasons, $\frac{y}{a}$ is termed the hyperbolic sine, or the hypo-sine of ϕ. This is written for brevity as $\sinh \phi$.

The value of $\sinh \phi$ can easily be obtained.

$$\sinh \phi = \frac{y}{a} = \sqrt{\frac{x^2}{a^2} - 1} = \sqrt{\cosh^2 \phi - 1}$$

$$= \sqrt{\frac{\epsilon^{2\phi} + 2 + \epsilon^{-2\phi}}{4} - 1}$$

$$= \sqrt{\frac{\epsilon^{2\phi} - 2 + \epsilon^{-2\phi}}{4}} = \frac{\epsilon^{\phi} - \epsilon^{-\phi}}{2}.$$

$$\therefore \sinh \phi = \frac{\epsilon^{\phi} - \epsilon^{-\phi}}{2}.$$

This result should be compared with the exponential value of $\sin \theta$. (See page 80.)

From the above we have—

$$\cosh^2 \phi - \sinh^2 \phi = 1,$$

$$\cosh \phi + \sinh \phi = \epsilon^{\phi},$$

$$\cosh \phi - \sinh \phi = \epsilon^{-\phi}.$$

The following important relations may now be determined:—

$$\cosh(\phi + \psi) = \frac{\epsilon^{(\phi+\psi)} + \epsilon^{-(\phi+\psi)}}{2}$$

$$= \frac{\epsilon^{\phi}.\epsilon^{\psi} + \epsilon^{-\phi}.\epsilon^{-\psi}}{2}$$

$$= \tfrac{1}{2}\{(\cosh \phi + \sinh \phi)(\cosh \psi + \sinh \psi) + (\cosh \phi - \sinh \phi)(\cosh \psi - \sinh \psi)\}$$

$$= \cosh \phi \cosh \psi + \sinh \phi \sinh \psi.$$

In particular,

$$\cosh 2\phi = \cosh^2 \phi + \sinh^2 \phi.$$

Also,

$$\sinh(\phi + \psi) = \frac{\epsilon^{(\phi+\psi)} - \epsilon^{-(\phi+\psi)}}{2} = \frac{\epsilon^{\phi}.\epsilon^{\psi} - \epsilon^{-\phi}.\epsilon^{-\psi}}{2}$$

$$= \tfrac{1}{2}\{(\cosh \phi + \sinh \phi)(\cosh \psi + \sinh \psi) - (\cosh \phi - \sinh \phi)(\cosh \psi - \sinh \psi)\}$$

$$= \cosh \phi \sinh \psi + \sinh \phi \cosh \psi.$$

In particular,

$$\sinh 2\phi = 2\sinh \phi \cosh \phi.$$

Also, if
$$y = \sinh a\phi = \frac{\epsilon^{+a\phi} - \epsilon^{-a\phi}}{2},$$

$$\frac{dy}{d\phi} = a\frac{\epsilon^{a\phi} + \epsilon^{-a\phi}}{2} = a \cosh a\phi,$$

$$\frac{d^2y}{d\phi^2} = a^2\frac{\epsilon^{a\phi} - \epsilon^{-a\phi}}{2} = a^2 \sinh a\phi.$$

And if
$$y = \cosh a\phi = \frac{\epsilon^{a\phi} + \epsilon^{-a\phi}}{2},$$

$$\frac{dy}{d\phi} = a\frac{\epsilon^{a\phi} - \epsilon^{-a\phi}}{2} = a \sinh a\phi,$$

$$\frac{d^2y}{d\phi^2} = a^2\frac{\epsilon^{a\phi} + \epsilon^{a\phi}}{2} = a^2 \cosh a\phi.$$

The importance of these hyperbolic functions depends on the above properties. Thus, if we have a differential equation

$$\frac{d^2y}{dx^2} - a^2y = 0,$$

we can write down the solution at once as

$$y = \mathsf{A} \cosh ax + \mathsf{B} \sinh ax,$$

A and B being two arbitrary constants introduced for the reason explained on page 203. The fact that this solution satisfies the given differential equation may be proved by substitution.

It should also be noticed that the equation to the catenary may be written in the form

$$y = c \cosh \frac{x}{c}.$$

An important relation between ordinary and hyperbolic sines and cosines may be easily determined.

$$\cos \theta = \frac{\epsilon^{\iota\theta} + \epsilon^{-\iota\theta}}{2} \text{ (page 80)},$$

where $\iota = \sqrt{-1}$.

For θ write $\iota\theta$.

$$\cos \iota\theta = \frac{\epsilon^{\iota^2\theta} + \epsilon^{-\iota^2\theta}}{2} = \frac{\epsilon^{-\theta} + \epsilon^{+\theta}}{2} = \cosh \theta.$$

$$\therefore \cosh \theta = \cos(\iota\theta).$$

Also, $$\sin \theta = \frac{\epsilon^{\iota\theta} - \epsilon^{-\iota\theta}}{2\iota} \text{ (page 80).}$$

For θ write $\iota\theta$.

$$\sin \iota\theta = \frac{\epsilon^{\iota^2\theta} - \epsilon^{-\iota^2\theta}}{2\iota} = \frac{\epsilon^{-\theta} - \epsilon^{+\theta}}{2\iota} = \frac{-\sinh \theta}{\iota} = \iota \sinh \theta.$$

$$\therefore \sinh \theta = -\iota \sin(\iota\theta).$$

Exercises.—(1.) Draw the curves $y = \cosh x$ and $y = \sinh x$ for values of x between -10 and $+10$.

(*Suggestion.*—Note that $\log_{10}(\epsilon^x) = x \log_{10}\epsilon$; the value of ϵ is given on page 24. Take $x = \cdot 5, 1, 2, 3 \ldots . 10$.)

The following points about these curves should be carefully noted :—

(a) The curve $y = \cosh x$ lies wholly above the axis of x, and is symmetrical with respect to the axis of y. When $x = 0$, $\cosh x = 1$.

(b) The curve $y = \sinh x$ lies, for positive values of x, in the $(+x, +y)$ quadrant; for negative values of x, it lies in the $(-x, -y)$ quadrant. When $x = 0$, $\sinh x = 0$.

(2.) If $y = \tanh x = \dfrac{\sinh x}{\cosh x}$, prove that $\dfrac{dy}{dx} = \dfrac{1}{\cosh^2 x}$.

(3.) Prove that—

(1) $\cos(\theta + \iota\phi) = \cos\theta \cosh\phi - \iota \sin\theta \sinh\phi$,

(2) $\sin(\theta + \iota\phi) = \sin\theta \cosh\phi + \iota \cos\theta \sinh\phi$,

(3) $\cos(\theta - \iota\phi) = \cos\theta \cosh\phi + \iota \sin\theta \sinh\phi$,

(4) $\sin(\theta - \iota\phi) = \sin\theta \cosh\phi - \iota \cos\theta \sinh\phi$.

(4.) Prove that $\sin^{-1}\dfrac{x}{a} = -\iota \log_e(\iota x + \sqrt{a^2 - x^2})$.

Suggestion.— $$\int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \sin^{-1}\frac{x}{a}.$$

This integral may also be written—

$$\int \frac{dx}{\{a^2+(\iota x)^2\}^{\frac{1}{2}}} = \frac{1}{\iota} \log_\epsilon \{\iota x + \sqrt{a^2 - x^2}\}.$$

(5.) Prove that $\tan^{-1} \dfrac{x}{a} = \iota \log_\epsilon \dfrac{a - \iota x}{a + \iota x}$.

Suggestion.— $\displaystyle\int \frac{dx}{a^2+x^2}$ may also be written $\displaystyle\int \frac{dx}{a^2 - (\iota x)^2}$.

(6.) Prove that

$$\int \frac{dx}{a+b \cosh x} = \frac{2}{\sqrt{b^2-a^2}} \tan^{-1} \left(\sqrt{\frac{b-a}{b+a}} \tanh \frac{x}{2} \right), \text{ if } b > a.$$

(Compare page 127.)

Suggestion.—Remembering that $\cosh x = \cosh^2 \dfrac{x}{2} + \sinh^2 \dfrac{x}{2}$, and that $1 = \cosh^2 \dfrac{x}{2} - \sinh^2 \dfrac{x}{2}$, the given integral may be written—

$$\int \frac{dx}{(a+b) \cosh^2 \frac{x}{2} + (b-a) \sinh^2 \frac{x}{2}}$$

$$= \frac{1}{(b-a)} \int \frac{\dfrac{dx}{\cosh^2 \frac{x}{2}}}{\dfrac{a+b}{b-a} + \tanh^2 \dfrac{x}{2}}.$$

Put

$$\tanh \frac{x}{2} = z,$$

$$\frac{1}{2} \frac{dx}{\cosh^2 \frac{x}{2}} = dz,$$

and the integral reduces to

$$\frac{2}{b-a} \int \frac{dz}{\dfrac{a+b}{b-a} + z^2}$$

$$= \frac{2}{b-a} \cdot \sqrt{\frac{b-a}{b+a}} \cdot \tan^{-1} \frac{z}{\sqrt{\dfrac{b+a}{b-a}}}. \quad \text{(See p. 94.)}$$

(7.) If, in the foregoing exercise, $b < a$, show that the value of the integral is

$$\frac{1}{\sqrt{a^2-b^2}} \log_e \frac{\sqrt{a-b} \cdot \tanh \frac{x}{2} - \sqrt{a+b}}{\sqrt{a-b} \cdot \tanh \frac{x}{2} + \sqrt{a+b}}.$$

Note.—For oral purposes, $\cosh \phi$ is pronounced as it is spelt; $\sinh \phi$ and $\tanh \phi$ are respectively pronounced *shin* ϕ and *tank* ϕ.

Problem.—A certain curve possesses the property that the length of that part of the tangent which lies between the curve and the axis of x is constant for all points on the curve. Find the equation to the curve.

Let AB (Fig. 64) be part of the curve, and let AD be that part

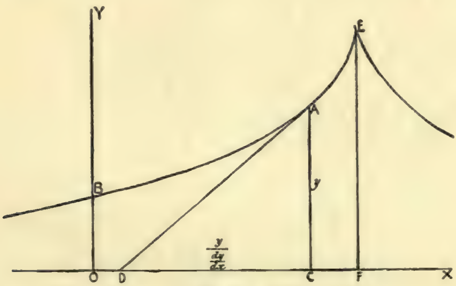

FIG. 64.

of the tangent of the curve at the point A which lies between the curve and the axis of x. Then the curve must be such that AD will have the same value from whatever point on the curve the tangent may be drawn.

Draw the ordinate $\mathrm{AC} = y$. Then $\frac{dy}{dx}$ will be equal to tan ADC.

$$\therefore \frac{y}{\mathrm{DC}} = \frac{dy}{dx}.$$

$$\therefore \mathrm{DC} = \frac{y}{\frac{dy}{dx}}.$$

Let the constant length, AD, be equal to c.

$$c^2 = \mathrm{DC}^2 + \mathrm{CA}^2 = \left\{\left(\frac{y}{\frac{dy}{dx}}\right)^2 + y^2\right\} = y^2\left\{\frac{1+\left(\frac{dy}{dx}\right)^2}{\left(\frac{dy}{dx}\right)^2}\right\}.$$

$$\therefore\ c = \frac{y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}}{\frac{dy}{dx}}.$$

This is the differential equation to the curve AB. To solve this equation we must first find the value of $\frac{dy}{dx}$ in terms of y and c.

$$(c^2 - y^2)\left(\frac{dy}{dx}\right)^2 = y^2.$$

$$\therefore\ \frac{dy}{dx} = \pm\frac{y}{\sqrt{c^2 - y^2}}.$$

The + sign will correspond to the branch of the curve which slopes upwards from left to right. Using this sign, transforming, and integrating, we get

$$\int dx = \int \frac{(c^2 - y^2)^{\frac{1}{2}}}{y}\,dy.$$

To integrate the quantity on the right-hand side of this equation, multiply both numerator and denominator by $\sqrt{c^2 - y^2}$.

$$\int \frac{(c^2 - y^2)}{y(c^2 - y^2)^{\frac{1}{2}}}\,dy = c^2\int\frac{dy}{y(c^2 - y^2)^{\frac{1}{2}}} - \int\frac{y\,dy}{(c^2 - y^2)^{\frac{1}{2}}}.$$

Now $$\int\frac{y\,dy}{(c^2 - y^2)^{\frac{1}{2}}} = -(c^2 - y^2)^{\frac{1}{2}}.$$

To find the value of $c^2\int\frac{dy}{y(c^2 - y^2)^{\frac{1}{2}}}$, substitute $y = \frac{1}{z}$.

$$\therefore\ dy = -\frac{dz}{z^2}.$$

$$c^2\int\frac{dy}{y(c^2-y^2)^{\frac{1}{2}}}=c^2\int\frac{-\frac{dz}{z^2}}{\frac{1}{z}\left(c^2-\frac{1}{z^2}\right)^{\frac{1}{2}}}$$

$$=-c^2\int\frac{\frac{dz}{z^2}}{\frac{1}{z^2}(c^2z^2-1)^{\frac{1}{2}}}=-c^2\int\frac{dz}{(c^2z^2-1)^{\frac{1}{2}}}$$

$$=-c^2\int\frac{dz}{c\left(z^2-\frac{1}{c^2}\right)^{\frac{1}{2}}}=-c\log_\epsilon\left\{z+\sqrt{z^2-\frac{1}{c^2}}\right\}$$

(see page 94)

$$=-c\log_\epsilon\left\{\frac{1}{y}+\sqrt{\frac{1}{y^2}-\frac{1}{c^2}}\right\}=-c\log_\epsilon\left\{\frac{c+\sqrt{c^2-y^2}}{cy}\right\}$$

$$=-c\log_\epsilon\frac{c+\sqrt{c^2-y^2}}{y}+c\log_\epsilon c.$$

Hence, finally,

$$\int dx=x=-c\log_\epsilon\frac{c+\sqrt{c^2-y^2}}{y}+(c^2-y^2)^{\frac{1}{2}}+\mathsf{A},$$

where A is an arbitrary constant, in which is included the constant quantity $c\log_\epsilon c$.

To find the value of A, let the tangent to the curve be perpendicular to the axis of x, at a point E corresponding to the abscissa $\mathsf{OF}=a$. Then when $x=a$, $y=c$.

$$a=-c\log_\epsilon\frac{c}{c}+\mathsf{A}$$

$$=c\log_\epsilon 1+\mathsf{A}=0+\mathsf{A}.$$

$$\therefore\ a=\mathsf{A}.$$

Therefore the equation to the curve up to the point F—

$$x=-c\log_\epsilon\frac{c+\sqrt{c^2-y^2}}{y}+(c^2-y^2)^{\frac{1}{2}}+a.$$

Beyond the point F it can easily be seen that $\frac{dy}{dx}$ will be a negative quantity, the curve sloping downwards from left to right. In that case we must take

$$\frac{dy}{dx} = -\frac{y}{\sqrt{(c^2 - y^2)}}.$$

The equation to this part of the curve will then be

$$x = +c \log_e \frac{c + \sqrt{c^2 - y^2}}{y} - (c^2 - y^2)^{\frac{1}{2}} + a.$$

The curve of which we have found the equation is termed the *tractory.** Its name is derived from a simple method by which the curve can be drawn.

Take a penknife, open the large blade fully, and set the small blade at a convenient angle to the handle (Fig. 65). Place the point

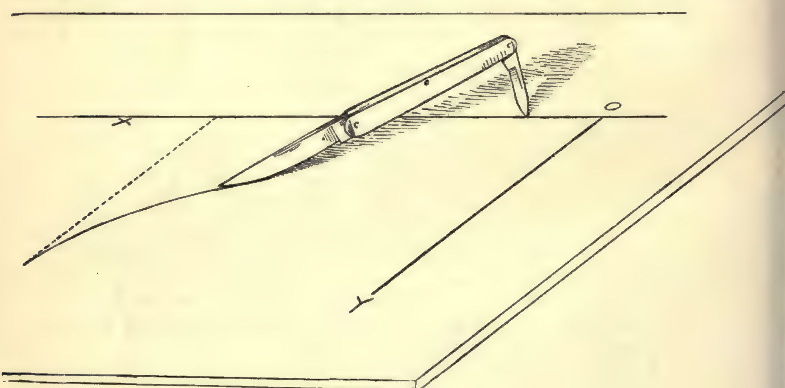

FIG. 65.

of the small blade on the point $x = a$, $y = 0$, and set the edge of the large blade on a straight line drawn through $x = a$, perpendicular to the axis of x. Then if the edge of the large blade be pressed firmly on to the paper, and the point of the small blade be moved to the right or left along the axis of x, the edge of the large blade will cut a curve into the paper as it is dragged along; and if the point of the small blade is in the plane of the cutting edge of the large blade, it can easily be seen that the line joining the cutting

* Lat. *traho, traxi, tractum,* to draw, drag along.

edge of the large blade and the point of the small blade will be a tangent to the curve, and that the distance along this tangent from the curve to the axis of x is constant.

Problem. — A uniform heavy beam is supported near its extremities and loaded at its middle point. It is required to determine the equation of the curve into which the beam is bent.

Let Fig. 66 represent the beam, supported at A and B, points

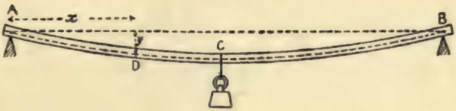

FIG. 66.

near its ends, and loaded at its centre point C with a weight of W lbs. Let w be the weight of unit length of the beam, and let l be the distance between the points of support.

Consider the reactions transmitted across an imaginary cross section of the beam at D. Since the two portions into which the beam is divided by this cross section are in equilibrium, the external forces acting on the portion AD must be in equilibrium with the forces called into play across the section at D by the bending of the beam.

The external forces acting on the portion AD of the beam comprise (1) the reaction of the support A, (2) the pull exerted by gravity on each element of the portion AD of the beam.

If the points A, B are very near to the ends of the beam, the total weight of the beam will be approximately equal to wl. Since the beam is uniform, and is loaded only at the centre point, the reaction of either support must be equal to $\frac{wl}{2}+\frac{W}{2}\cdot$

Let the distance $AD = x$. Then the moment of the reaction at the point A, about the point D, will be equal to $\left(\frac{wl}{2}+\frac{W}{2}\right)x$. The couple, of which this is the moment, tends to rotate the cross section at D about a horizontal axis in the sense in which the hands of a clock revolve. If we agree to affix a + sign to those couples which tend to rotate the section at D in the sense in which the hands of a clock revolve, the above couple will have a + sign.

In addition to the reaction at the point A, each unit length of AD will be pulled downwards with a force w. If we take a small length $d\xi$, at a distance ξ to the left of D, the force acting on this small length will be equal to $wd\xi$, and the moment of this force about D will be equal to $w\xi d\xi$. The sum of the moments of all such forces, acting on the elements into which AD may be supposed to be divided, will be equal to

$$w\int_0^x \xi d\xi = w\frac{x^2}{2}.$$

Each of these moments must be given a $-$ sign, since the corresponding couple tends to rotate the section at D in a sense opposite to that in which the hands of a clock revolve. Hence the sum of the moments about the point D, of the external forces acting on AD, is equal to

$$\left(\frac{wl+\mathrm{W}}{2}\right)x - \frac{wx^2}{2}.$$

This is termed the *bending moment* at the point D.

We must now obtain an expression for the resultant of the forces called into play across the section at D by the bending of the beam. It is easily seen that near the upper surface of the beam the fibres will be compressed longitudinally, whilst near the lower surface of the beam the fibres will be elongated. At some intermediate position the fibres will be neither elongated nor compressed. We may use the term *neutral surface* to designate the infinitely thin layer, extending from end to end of the beam, in which the fibres are neither elongated nor compressed.

The compression of the fibres near the upper surface, and the elongation of the fibres near the lower surface of the beam, will call into play a number of couples, tending to rotate the section at D in a sense opposite to that in which the hands of a clock revolve. The sum of the moments of these couples must be equal in magnitude, but opposite in sign, to the bending moment at D.

Let *acbd* (Fig. 67) represent the cross-section of the beam at D. Let *ab* be the section of the layer which is neither compressed nor elongated. Above *ab* the fibres of the beam are compressed, below *ab* they are elongated. Let f be the restoring force *per unit area*, at a distance z from the line *ab*. We may give z a $+$ sign for points above, and a $-$ sign for points below *ab*. Let ds be a small element of area, at a distance z above ab. Then the moment of the restoring force called into play by the compression of the fibres passing through ds will be equal to $fds \times z = fzds$. f will vary from point to point, being greatest near the surface of the beam, and equal to zero in the section ab. We shall now find the value of f.

Let FDEG (Fig. 67) represent a longitudinal section of the

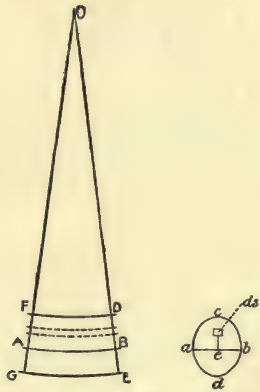

FIG. 67.

beam near D, and let AB be the section of the neutral surface. Then, if AB is short, we may treat it as a part of a circle of radius OB. Then O will be the centre of curvature, and OB will be the radius of curvature of AB (see page 84). Let $OB = R$. Then, if the angle $AOB = d\theta$, we have

$$\text{Length of } AB = Rd\theta.$$

If the beam were not bent, FD and GE would both be equal in length to AB. But, for the reasons already explained, fibres above AB are shortened, and those below AB are lengthened. At a distance z above AB the length of a fibre will be

$$(R - z)d\theta.$$

The length of a fibre below AB is found by substituting $-z$ for z in this expression.

Let Y be Young's modulus of the substance of which the beam is composed. Then, by definition,

$$\mathsf{Y}=\frac{\text{restoring force (per unit area)}}{\text{compression (or extension) per unit length}}.$$

At a distance z above AB the compression $=\mathsf{R}d\theta-(\mathsf{R}-z)d\theta=zd\theta$. The original length of the fibre was $\mathsf{R}d\theta$.

$\therefore$ compression, per unit length, at a distance z above AB,

$$=\frac{zd\theta}{\mathsf{R}d\theta}=\frac{z}{\mathsf{R}}.$$

The extension at a distance z below AB will be given by substituting $-z$ for z in this expression.

Hence, remembering that f indicates the restoring force per unit area, at a point at a distance z above the section of the neutral surface, we have

$$\mathsf{Y}=\frac{f}{\frac{z}{\mathsf{R}}}.$$

$$\therefore f=\frac{z}{\mathsf{R}}\mathsf{Y}.$$

Hence the force acting upwards through the area ds (Fig. 67), at a distance z above the line ab, will be equal to $\mathsf{Y}\frac{z}{\mathsf{R}}ds$.

Let us apply two forces, each numerically equal to $\mathsf{Y}\frac{z}{\mathsf{R}}ds$, at the point e (Fig. 67), one force acting upwards through the paper, and the other acting in an opposite direction. Then the force $\mathsf{Y}\frac{z}{\mathsf{R}}ds$ acting upwards through ds, and the equal force acting downwards at e, will constitute a couple, of which the moment is $\mathsf{Y}\frac{z}{\mathsf{R}}ds\times z=\mathsf{Y}\frac{z^2}{\mathsf{R}}ds$. We can therefore replace the force $\mathsf{Y}\frac{z}{\mathsf{R}}ds$, acting upwards through ds, by a couple $\mathsf{Y}\frac{z^2}{\mathsf{R}}ds$, and a force equal to $\mathsf{Y}\frac{z}{\mathsf{R}}ds$, acting upwards at e. If the whole of the area $acbd$ be divided into small elements of area similar to ds, and if the process just indicated be carried out with respect to each, then the forces called into play over the surface *acbd* may be replaced by a resultant force equal to

$$\frac{\mathsf{Y}}{\mathsf{R}}\int z ds,$$

acting along the neutral surface, at right angles to the line *ab*, and a couple equal in magnitude to

$$\frac{\mathsf{Y}}{\mathsf{R}}\int z^2 ds,$$

both of the integrations being extended over the whole of the surface *acbd*.

Now the resultant force acting along the neutral surface, at right angles to the line *ab*, must be equal to zero; since, if this were not so, the fibres in the neutral surface would be either compressed or elongated.

Hence $$\frac{\mathsf{Y}}{\mathsf{R}}\int z ds = 0. \quad \therefore \int z ds = 0.$$

But this is the condition that the centre of gravity of the surface *acbd* should lie on the line *ab*.

The line passing through the centres of gravity of successive sections of the beam is termed the neutral axis of the beam. The neutral axis will obviously lie in the neutral surface.

The resultant couple, equal in magnitude to

$$\frac{\mathsf{Y}}{\mathsf{R}}\int z^2 ds,$$

must be equal in magnitude, but opposite in sign, to the bending movement at the point D (Fig. 66). Also,

$$\int z^2 ds,$$

the integration being performed over the whole of the surface *acbd*, is equal to the moment of inertia of that surface about the line *ab*—an element of surface being substituted for an element of mass in the ordinary definition of a moment of inertia. (See page 152.)

Let $$\int z^2 ds = \mathbf{I}.$$

Then $$-\frac{\mathsf{Y}}{\mathsf{R}}\mathbf{I} = \left(\frac{wl + \mathsf{W}}{2}\right)x - \frac{wx^2}{2}.$$ (See page 222.)

But, as proved on pages 84 to 87, the radius of curvature at a point x, y on a curve is equal to

$$\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

In the present case, if the beam is not very greatly bent, $\frac{dy}{dx}$ will be small at all points on the curve, and $\left(\frac{dy}{dx}\right)^2$ may be neglected in comparison with 1. Hence

$$R=\frac{1}{\frac{d^2y}{dx}}.$$

$$\therefore\ -YI\frac{d^2y}{dx^2}=\left(\frac{wl+W}{2}\right)x-\frac{wx^2}{2}.$$

This is the differential equation to the curve given by a longitudinal section of the neutral surface, or the curve in which the neutral axis lies.

The first integral of this equation is obviously

$$-YI\int\frac{d^2y}{dx^2}\,dx=\frac{wl+W}{2}\int xdx-\frac{w}{2}\int x^2dx,$$

or

$$-YI\frac{dy}{dx}=\frac{wl+W}{2}\cdot\frac{x^2}{2}-\frac{w}{2}\cdot\frac{x^3}{3}+A,$$

where A is an arbitrary constant.

To find the value of A, notice that $\frac{dy}{dx}=0$ at the centre of the beam—that is, when $x=\frac{l}{2}$.

$$\therefore\ 0=\frac{wl+W}{2}\cdot\frac{l^2}{8}-\frac{w}{6}\cdot\frac{l^3}{8}+A.$$

$$\therefore\ A=-wl^3\left(\frac{1}{16}-\frac{1}{48}\right)-\frac{wl^2}{16}.$$

$$\therefore\ A=-\frac{wl^3}{24}-\frac{wl^2}{16}.$$

Performing the second integration, we get

$$-\mathrm{YI}\int\frac{dy}{dx}\,dx=\frac{wl+\mathrm{W}}{4}\int x^2dx-\frac{w}{6}\int x^3dx+\mathrm{A}\int dx$$

$$-\mathrm{YI}y=\frac{wl+\mathrm{W}}{4}\cdot\frac{x^3}{3}-\frac{w}{6}\frac{x^4}{4}+\mathrm{A}x+\mathrm{B}.$$

To find the value of the arbitrary constant B, notice that when $x=0$, $y=0$.

$$\therefore\ 0=\mathrm{B}.$$

Hence, substituting this value of B, and the value already obtained of A, we get

$$\mathrm{YI}y=\frac{w}{24}x^4-\frac{wl+\mathrm{W}}{12}x^3+\left(\frac{wl^3}{24}+\frac{\mathrm{W}l^2}{16}\right)x.$$

As a particular case, take $\mathrm{W}=0$.

$$\mathrm{YI}y=\frac{w}{24}x^4-\frac{wl}{12}x^2+\frac{wl^3}{24}x$$

$$=\frac{wx}{24}(x^3-2lx^2+l^3).$$

This is the equation to the unloaded beam.

If the beam is supposed to be loaded at its centre point with a weight W, which is very great in comparison with w, we may put $w=0$. Then

$$\mathrm{YI}y=\frac{\mathrm{W}l^2}{16}x-\frac{\mathrm{W}x^2}{12}=\frac{\mathrm{W}x}{4}\cdot\left(\frac{l^2}{4}-\frac{x^2}{3}\right).$$

RECTANGULAR BEAM.—If the beam be rectangular, of breadth B, and depth D, we have (see page 152)

$$\mathrm{I}=\int z^2ds=\int_{-\frac{\mathrm{D}}{2}}^{+\frac{\mathrm{D}}{2}}\mathrm{B}z^2dz=\left[\mathrm{B}\frac{z^3}{3}\right]_{-\frac{\mathrm{D}}{2}}^{+\frac{\mathrm{D}}{2}}$$

$$=\frac{\mathrm{BD}^3}{12}.$$

Let d be the deflection of the beam at its middle point when loaded there with a weight W. Then d will be equal to the value of y when $x=\frac{l}{2}$.

$$\mathsf{Y}\frac{\mathsf{BD}^3}{12}\cdot d\cdot=\frac{w}{24}\cdot\frac{l^4}{16}-\frac{wl+\mathsf{W}}{12}\cdot\frac{l^3}{8}+\left(\frac{wl^3}{24}+\frac{\mathsf{W}l^2}{16}\right)\frac{l}{2}$$

$$=wl^4\frac{5}{384}-\mathsf{W}l^3\cdot\frac{1}{48}.$$

Let the deflection of the beam at its middle point be increased from d to $d+\delta$, when W is increased to W + 1. Then

$$\mathsf{Y}\frac{\mathsf{BD}^3}{12}(d+\delta)=wl^4\frac{5}{384}-(\mathsf{W}+1)l^3\frac{1}{48}.$$

$$\therefore\ \mathsf{Y}\frac{\mathsf{BD}^3\delta}{12}=\frac{l^3}{48}.$$

$$\therefore\ \mathsf{Y}=\frac{l^3}{4\mathsf{BD}^3\delta}.$$

This gives us a method of determining the Young's modulus of the substance of which a beam is composed. We measure δ, the increase in the deflection of the centre point of the beam, per unit increase in the load, and determine l, B, and D by direct measurement.

The equation

$$\delta=\frac{l^3}{4\mathsf{BD}^3\mathsf{Y}}$$

shows that the deflection of the centre point of the beam, per unit increase of load, is proportional to the cube of the length of the beam, and inversely proportional to the product of the breadth into the cube of the depth of the beam.

Problem.—A heavy body is hung from a fixed support by means of an elastic fibre. Being displaced vertically from its position of equilibrium, and then released, it oscillates in a vertical straight line about that position. It is required to determine the equation to the motion of the body.

FIG. 68.

Let B (Fig. 68) be the position of equilibrium of the body, and let the initial displacement be BC. It is required to determine an equation, giving the position of the body with respect to the point B at any time t seconds after its release.

Let the mass of the body be m. Then gravity is pulling this downwards with a constant force mg, where g is the acceleration due to gravity.

(If m is measured in lbs., g will be equal to 32 ft./sec.^{2}, and all distances must be measured in feet. If m is measured in grammes, g will be equal to 981 cm./sec.^{2}, and all distances must be measured in centimetres. In both cases time is measured in seconds.)

When the body is at B, its position of equilibrium, the force which gravity exerts on it is just neutralized by the upward pull of the stretched elastic fibre. When the body is below the point B, there will be a resultant upward pull equal to the increase in the tension of the fibre. When the body is above B, there will be a resultant downward pull equal to the diminution in the tension of the fibre.

By Hooke's law, the increase in the tension of an elastic filament is proportional to the increase in the length of the filament.

Let f be the force necessary to increase the length of the filament by unity (one centimetre or one foot, according to the system of units used). Then, if the body is x units of length below B, the resultant upward force will be equal to fx.

If the body is at a distance x *above* B, the resultant force acting on it will be $-fx$, the negative sign indicating that the pull is downwards. At B there will be no resultant force on the body.

Let x be measured vertically downwards. The velocity of the body downward, at a time t, will be equal to $\frac{dx}{dt}$, and the acceleration in the same direction will be equal to $\frac{d^2x}{dt^2}$. Hence, when the body is below B, the acceleration of the body *toward* B will be equal to $-\frac{d^2x}{dt^2}$; and since *force* = *mass* × *acceleration*, the force pulling the body toward B will be equal to $-m\frac{d^2x}{dt^2}$. This force is equal to the pull of the elastic filament, the value of which has already been found. Hence

$$-m\frac{d^2x}{dt^2} = fx$$

is the *differential equation* to the motion of the body.

To solve this equation, multiply both sides by $\frac{dx}{dt}dt$, and integrate. Then

$$-m\int\frac{dx}{dt}\frac{d^2x}{dt^2}dt=f\int x\frac{dx}{dt}dt$$

$$-\frac{m}{2}\cdot\left(\frac{dx}{dt}\right)^2=f\frac{x^2}{2}-\mathrm{C},$$

where we add the arbitrary constant $-\mathrm{C}$ to obtain the most general result.

This equation may be written

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2+f\frac{x^2}{2}=\mathrm{C} \quad . \quad . \quad . \quad (1).$$

Now $$\frac{m}{2}\left(\frac{dx}{dt}\right)^2=\frac{m}{2}\times(\text{velocity})^2$$

= the kinetic energy of the body at the time t.

$$f\frac{x^2}{2}=f\int xdx$$

= the work done in displacing the body through a distance x from its position of equilibrium. In other words, this quantity equals the potential energy of the body at the time t. Hence, expressed in words, (1) signifies that the sum of the potential and kinetic energies of the body remains constant. This has been tacitly assumed in writing down the differential equation to the body's motion, since no account was taken of any frictional forces by which the energy of the body might be dissipated.

To find the value of C, let a be the distance below B to which the body was primarily displaced. At the instant of release, when the displacement was a, the velocity of the body was equal to zero. Hence

$$0=f\frac{a^2}{2}-\mathrm{C}. \quad \therefore\ \mathrm{C}=f\frac{a^2}{2}.$$

This shows that the sum of the potential and kinetic energies of the body remains equal to the potential energy of the body at the moment of its release.

We may now rewrite (1)—

$$\left(\frac{dx}{dt}\right)^2 = \frac{f}{m}(a^2 - x^2).$$

$$\therefore \frac{dx}{dt} = \sqrt{\frac{f}{m}} \cdot \sqrt{a^2 - x^2}.$$

$$\therefore \int \frac{dx}{(a^2 - x^2)^{\frac{1}{2}}} = \sqrt{\frac{f}{m}} \cdot \int dt.$$

$$\sin^{-1}\frac{x}{a} = \sqrt{\frac{f}{m}} \cdot t + \mathsf{B},$$

where B is an arbitrary constant. To find the value of B, notice that at the instant when the body is released, and $t = 0$, $x = a$.

$$\therefore \sin^{-1}\frac{a}{a} = \sin^{-1} 1 = \mathsf{B}.$$

But $\sin^{-1} 1 =$ the angle of which the sine is equal to 1. Hence $\sin^{-1} = \frac{\pi}{2}$.

$$\therefore \mathsf{B} = \frac{\pi}{2}.$$

$$\therefore \sin^{-1}\frac{x}{a} = \sqrt{\frac{f}{m}} \cdot t + \frac{\pi}{2}.$$

$$\therefore \frac{x}{a} = \sin\left\{\sqrt{\frac{f}{m}} \cdot t + \frac{\pi}{2}\right\}$$

$$= \cos\left(\sqrt{\frac{f}{m}}\, t\right).$$

$$\therefore x = a \cos\left(\sqrt{\frac{f}{m}}\, t\right).$$

The meaning of this equation should now be carefully studied.

(1.) When $\quad t = 0, \cos\left(\sqrt{\frac{f}{m}} \cdot t\right) = \cos 0 = 1.$

$$\therefore x = a.$$

(2.) When $\sqrt{\frac{f}{m}} \cdot t = \frac{\pi}{2}, \cos\left(\sqrt{\frac{f}{m}} \cdot t\right) = \cos\frac{\pi}{2} = 0.$

$$\therefore x = 0.$$

Hence, in a time equal to $t=\frac{\pi}{2}\sqrt{\frac{m}{f}}$, the body will pass through its position of equilibrium, moving upwards.

(3.) When $\sqrt{\frac{f}{m}}\cdot t=\pi$, $\cos\left(\sqrt{\frac{f}{m}}\cdot t\right)=\cos\pi=-1$.

$$\therefore x=-a.$$

Hence, after a time $t=\pi\sqrt{\frac{m}{f}}$, the body will be at a position a units of length *above* its position of equilibrium.

(4.) When $\sqrt{\frac{f}{m}}t=\frac{3\pi}{2}$, $\cos\left(\sqrt{\frac{f}{m}}\cdot t\right)=\cos\frac{3\pi}{2}=0$.

Hence, after a time $t=\frac{3\pi}{2}\sqrt{\frac{m}{f}}$, the body will once more pass through its position of equilibrium, this time moving downwards.

(5.) When $\sqrt{\frac{f}{m}}t=2\pi$, $\cos\left(\sqrt{\frac{f}{m}}\cdot t\right)=\cos 2\pi=1$. $\therefore x=a$.

Hence, after a time $t=2\pi\sqrt{\frac{m}{f}}=\mathsf{T}$ (say), the body will have regained its initial position, at a distance a below its point of equilibrium.

In other words, the time T required to complete an oscillation about the point of equilibrium $=2\pi\sqrt{\frac{m}{f}}$.

This is a most important result, and gives us one method of determining g, the acceleration due to gravity.

For, suppose that we find that a mass M, added to the body, displaces the latter by a distance d.

Then a force $\mathsf{M}g$ produces a displacement d.

$$\therefore \text{ a force } \frac{\mathsf{M}g}{d} \text{ produces unit displacement.}$$

$$\therefore f=\frac{\mathsf{M}g}{d}.$$

$$\therefore \mathsf{T}=\text{time of one complete oscillation}=2\pi\sqrt{\frac{md}{\mathsf{M}g}}.$$

If T, the time of one complete oscillation, be calculated from the observed time occupied by a known number of oscillations, and m, d, and M are known, we have

$$g=\frac{4\pi^2 md}{\mathsf{M}t^2}.$$

Problem.—In the above, find the velocity with which the body passes through its position of equilibrium.

$$\text{Velocity}=\frac{dx}{dt}=-a\sqrt{\frac{f}{m}}\cdot\sin\left(\sqrt{\frac{f}{m}}\,t\right).$$

The body will pass upwards through its position of equilibrium, when $\sqrt{\frac{f}{m}}\cdot t=\frac{\pi}{2}$, and downwards through the same position, when $\sqrt{\frac{f}{m}}\cdot t=\frac{3\pi}{2}$.

In the first of these two cases,

$$\frac{dx}{dt}=-a\sqrt{\frac{f}{m}}\sin\frac{\pi}{2}=-a\sqrt{\frac{f}{m}}=\frac{-2\pi a}{\mathsf{T}}.$$

In the second case,

$$\frac{dx}{dt}=-a\sqrt{\frac{f}{m}}\sin\frac{3\pi}{2}=+a\sqrt{\frac{f}{m}}=\frac{2\pi a}{\mathsf{T}}.$$

The body will, of course, in the absence of frictional forces, continue moving upwards and downwards alternately, the time T elapsing between two successive passages of the body through any position, *in the same direction*, being equal to $2\pi\sqrt{\frac{m}{f}}$.

Exercise.—Since the solution of the differential equation

$$\frac{d^2y}{dx^2}-a^2y=0$$

is given by $\qquad y=\mathsf{A}'\cosh ax+\mathsf{B}'\sinh ax,$

show that the solution of the equation

$$\frac{d^2y}{dx^2}+a^2y=0$$

is given by $\qquad y=\mathsf{A}\cos ax+\mathsf{B}\sin ax=\mathsf{C}\sin(ax+\phi),$

where $\qquad \mathsf{C}=\sqrt{\mathsf{A}^2+\mathsf{B}^2}$, and $\tan\phi=\frac{\mathsf{A}}{\mathsf{B}}$.

APPENDIX.

TRIGONOMETRICAL RATIOS AND FORMULÆ.

Measurement of Angles.

1. Let CD, CE (Fig. 69) be any two straight lines intersecting

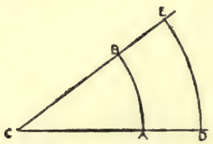

Fig. 69.

at C. If any circular arcs AB, DE be drawn with C as centre, then

$$\frac{\text{arc AB}}{\text{CA}}=\frac{\text{arc DE}}{\text{CD}}, \text{ etc.};$$

in other words, the ratio $\frac{\text{arc}}{\text{radius}}$ is constant, provided the arc subtends a constant angle at the centre.

This gives us a method of measuring angles. The ratio $\frac{\text{arc}}{\text{radius}}$ is termed the *circular measure* of the angle subtended by the arc at the centre of the circle.

Unit angle, termed a *radian*, corresponds to the relation

$$\frac{\text{arc}}{\text{radius}}=1.$$

Two right angles $=3{\cdot}14159$ radians $=\pi$ radians.

2. From any point C, in a straight line AB, describe a semicircle BDEA. Divide BDEA into 180 equal parts, and join the points so obtained to C by means of straight lines. Then the angle between any two consecutive lines is defined as one *degree* (1°). The angle at C, subtended by n equal parts of the circumference, will thus be equal to n°.

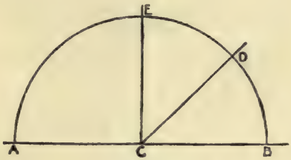

FIG. 70.

Each degree is subdivided in a similar manner into 60 equal angles, each such angle being termed a *minute* ($1'$). A minute is further divided into 60 equal parts, each part being termed a *second* ($1''$).

One radian $= 57^\circ\ 17'\ 45''$.

A movable radius may be supposed to start from the position CB (pointing toward the right), and to move so that its free end passes through all intermediate positions along the curve BDEA until the position A is reached. The radius is then said to have swept out a *positive* angle of two right angles, or π radians, or 180°. If the radius continues its motion in the same sense until it once more reaches the position CB, it will have swept out a positive angle of four right angles, or 2π radians, or 360°. In a similar manner, if the radius moves further in the same sense till the position CA is once more reached, a positive angle of six right angles, or 3π radians, or 540° will have been swept out. This process may be continued indefinitely, so that a radius may sweep out an angle of any magnitude whatever.

If the radius is rotated about the centre C in a sense opposite to that defined above—that is, if it rotates in a sense similar to that of the hands of a clock—then a *negative* angle is said to have been swept out. The numerical magnitude of the angle is measured in the manner described above.

Similar Triangles.

Let AB, AC be any two straight lines intersecting in A. From

D and F, points in the line AB, draw DE, FG perpendicular to AC; and from K, a point in AC, draw KH perpendicular to AB.

Then DEA, FGA, KHA are similar right-angled triangles. The lines ED, GF, HK are termed the *perpendiculars;* AE, AG, AH are termed the *bases;* and AD, AF, AK are termed the *hypotenuses* of these respective triangles.

Trigonometrical Ratios.

From a well-known property of similar triangles,

$$\frac{ED}{AE} = \frac{GF}{AG} = \frac{KH}{AH}.$$

The value of the ratio $\frac{\text{perpendicular}}{\text{base}}$ will vary only with the

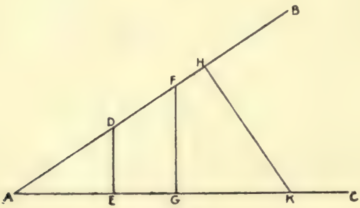

Fig. 71.

angle BAC; it is termed the *tangent of the angle* BAC. This is written, for brevity,

$$\frac{ED}{AE}, \text{ etc.} = \frac{\text{perpendicular}}{\text{base}} = \tan A.$$

The ratio of similar sides of these similar triangles will always be constant. For convenience of reference, definite names have been given to the various ratios which can be formed. Thus,

$$\frac{ED}{AD} = \frac{GF}{AF} = \frac{HK}{AK} = \frac{\text{perpendicular}}{\text{hypotenuse}}$$

$$= \text{sine of the angle BAC} = \sin A.$$

$$\frac{AE}{AD} = \frac{AG}{AF} = \frac{AH}{AK} = \frac{\text{base}}{\text{hypotenuse}}$$

$$= \text{cosine of the angle BAC} = \cos A.$$

$$\frac{AE}{ED}=\frac{AG}{GF}=\frac{AH}{HK}=\frac{\text{base}}{\text{perpendicular}}$$

$$=\text{cotangent of the angle } BAC=\cot A.$$

For convenience, the square of sin A, or $(\sin A)^2$, is generally written $\sin^2 A$.

Similarly, $$\cos^2 A=(\cos A)^2,$$

$$\tan^2 A=(\tan A)^2.$$

Particular Values of the Trigonometrical Ratios.

1. Let the angle A of the right-angled triangle BCA (Fig. 72)

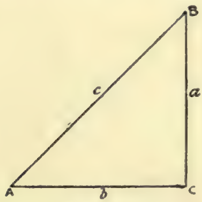

Fig. 72.

be equal to 45°, or $\frac{\pi}{4}$. Let us, for convenience, denote the length of the side BC, which is opposite to the angle A, by the small letter a. Similarly, for side AC we may write b, and for AB we may write c.

Let the length of the side b be equal to l. Then $a=l$. Further, $c^2=a^2+b^2=2l^2$. $\therefore\ c=l\sqrt{2}$.

$$\tan A=\tan\frac{\pi}{4}=\frac{a}{b}=\frac{l}{l}=1.$$

$$\cos A=\cos\frac{\pi}{4}=\frac{b}{c}=\frac{l}{l\sqrt{2}}=\frac{1}{\sqrt{2}}.$$

$$\sin A=\sin\frac{\pi}{4}=\frac{a}{c}=\frac{l}{l\sqrt{2}}=\frac{1}{\sqrt{2}}.$$

2. Let ABD (Fig. 73) be an equilateral triangle. Then $A=60^\circ=\frac{\pi}{3}$. Draw BC perpendicular to AD. Then $AC=\frac{1}{2}AD$.

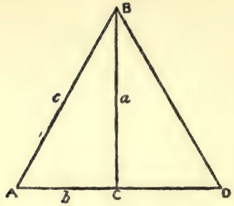

Fig. 73.

Let the length of a side of the equilateral triangle ABD be equal to l. Then, in the right-angled triangle ABC, $c = l$, $b = \frac{l}{2}$.

$$a^2 = c^2 - b^2 = l^2 - \frac{1}{4} l^2 = \frac{3l^2}{4}.$$

$$\therefore\ a = l \frac{\sqrt{3}}{2}.$$

$$\sin A = \sin 60^\circ = \sin \frac{\pi}{3} = \frac{l \frac{\sqrt{3}}{2}}{l} = \frac{\sqrt{3}}{2}.$$

$$\cos A = \cos 60^\circ = \cos \frac{\pi}{3} = \frac{\frac{l}{2}}{l} = \frac{1}{2}.$$

$$\tan A = \tan 60^\circ = \tan \frac{\pi}{3} = \frac{l \frac{\sqrt{3}}{2}}{\frac{l}{2}} = \sqrt{3}.$$

3. The angle ABC will be equal to 30°, or $\frac{\pi}{6}$. If we denote this angle by B, we find the following results:—

$$\sin B = \sin 30^\circ = \sin \frac{\pi}{6} = \frac{b}{c} = \frac{\frac{1}{2} l}{l} = \frac{1}{2}.$$

$$\cos B = \cos 30^\circ = \cos \frac{\pi}{6} = \frac{a}{c} = \frac{l \frac{\sqrt{3}}{2}}{l} = \frac{\sqrt{3}}{2}.$$

$$\tan B = \tan 30^\circ = \tan\frac{\pi}{6} = \frac{b}{a} = \frac{\frac{1}{2}l}{l\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}.$$

4. Let the angle A of the right-angled triangle ABC (Fig. 74)

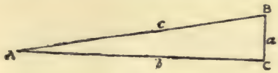

Fig. 74.

be very small. Then the side a will be very small, and will diminish as A is diminished. Also, the side b will become more and more nearly equal to c as A is diminished. Hence, as A approaches zero, a approximates to zero, and b approximates to c.

Then, if we understand, by the expression cos 0, the value to which the ratio $\frac{b}{c}$ approximates as A is made more and more nearly equal to zero, a similar meaning being given to the other ratios, we have—

$$\cos 0 = \frac{b}{c} = 1,$$

$$\sin 0 = \frac{a}{c} = 0,$$

$$\tan 0 = \frac{a}{b} = 0.$$

5. Let ABC (Fig. 75) be a right-angled triangle, of which the angle A is very nearly equal to 90°, or $\frac{\pi}{2}$ radians. Then, as A approaches $\frac{\pi}{2}$, b will approach the value zero, and a will become more and more nearly equal to c. Hence, if $\cos\frac{\pi}{2}$ is understood to mean the value to which the ratio $\frac{b}{c}$ approximates, as A is made more and more nearly equal to $\frac{\pi}{2}$, a similar meaning being given to the other ratios, we have—

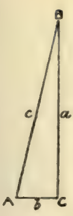

Fig. 75.

$$\cos\frac{\pi}{2}=\frac{b}{c}=0,$$

$$\sin\frac{\pi}{2}=\frac{a}{c}=1,$$

$$\tan\frac{\pi}{2}=\frac{a}{b}=\infty.$$

The meaning of the last result is, that as A is made more and more nearly equal to $\frac{\pi}{2}$, the value of tan A increases indefinitely; so that by making A very nearly equal to $\frac{\pi}{2}$, tan A can be made larger than any conceivable number.

6. The trigonometrical ratios for any angle can be determined graphically by drawing two lines intersecting each other at the required angle, dropping a perpendicular from a point in one on to the other line, and then measuring the three sides of the right-angled triangle so formed.

Special formulæ and series can also be used to calculate the trigonometrical ratios of an angle. Some of these will be developed subsequently.

Conventions as to Signs.

As already explained, an angle may have any magnitude whatever, and need not necessarily be less than two, or even four right angles. We can also express the trigonometrical ratios of an angle, however great that may be.

Let us suppose that a movable radius rotates so that its free end passes successively through the points A, B, F, C, G, D, H, E, A (Fig. 76). In order to determine the trigonometrical ratios for any angle swept out (say the angle COA), we drop a perpendicular from the free end C of the radius on to OA, or that line produced to G. The *magnitude* of the trigonometrical ratios for the angle COA will be found by taking Oc as the base, cC as the perpendicular, and OC as the hypotenuse in the general expressions. (See page 236.)

The following conventions as to signs are used:—

1. The length of the movable radius (which will always form the hypotenuse) is always counted as +.

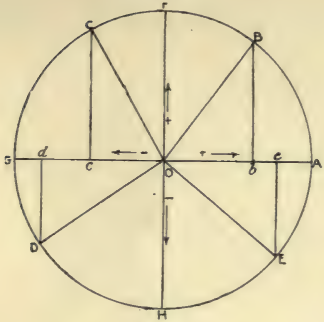

FIG. 76.

2. Distances measured to the right from O are +, and those measured to the left from O are −.

3. Distances measured upwards from points in the line GOA are +, and those measured downwards from points in the same line are −.

Thus, in obtaining the trigonometrical ratios of the angle COA, we must count Oc negative, cC positive, OC positive. Hence the sine of COA will be positive, the cosine of COA will be negative, and the tangent of COA will be negative.

The student should now verify the results given in the following table :—

Angle (θ).	Sin θ.	Cos θ.	Tan θ.	Cot θ.
$0<\theta<\frac{\pi}{2}$	+	+	+	+
$\frac{\pi}{2}<\theta<\pi$	+	−	−	−
$\pi<\theta<\frac{3\pi}{2}$	−	−	+	+
$\frac{3\pi}{2}<\theta<2\pi$	−	+	−	−

1. Let the angle $\mathsf{CO}c=\theta$; then the angle COA is equal to $\pi-\theta$. The angle $\mathsf{CO}c$ is termed the *supplement* of the angle COA.

Let the numerical magnitude of $\mathsf{O}c=l_1$, of $c\mathsf{C}=l_2$, and of $\mathsf{OC}=l_3$.

Then
$$\sin \mathsf{COA}=\sin(\pi-\theta)=+\frac{l_2}{l_3}=\sin\theta,$$
$$\cos \mathsf{COA}=\cos(\pi-\theta)=\frac{-l_1}{l_3}=-\cos\theta,$$
$$\tan \mathsf{COA}=\tan(\pi-\theta)=\frac{l_2}{-l_1}=-\tan\theta.$$

2. If the angle $\mathsf{DO}d=\theta$, then in moving from the position OA to OD, in the positive sense (see page 235), the radius has swept out an angle $\pi+\theta$. It can now be proved, by reasoning similar to that just used, that

$$\sin(\pi+\theta)=-\sin\theta,$$
$$\cos(\pi+\theta)=-\cos\theta,$$
$$\tan(\pi+\theta)=+\tan\theta.$$

3. If the angle $\mathsf{EO}e=\theta$, in moving from the position OA to OE, in the positive sense, an angle of $2\pi-\theta$ has been swept out. It can be proved, by reasoning similar to that just used, that

$$\sin(2\pi-\theta)=-\sin\theta,$$
$$\cos(2\pi-\theta)=+\cos\theta,$$
$$\tan(2\pi-\theta)=-\tan\theta.$$

4. If the movable radius has swept out the angle $\mathsf{AOE}=\theta$ *in the negative sense*, it may be proved, by reasoning similar to the above, that

$$\sin(-\theta)=-\sin\theta,$$
$$\cos(-\theta)=+\cos\theta,$$
$$\tan(-\theta)=-\tan\theta.$$

Trigonometrical Ratios of the Complement of an Angle.

In the right-angled triangle ABC (Fig. 77) the sum of the angles A and B is equal to $\frac{\pi}{2}$.

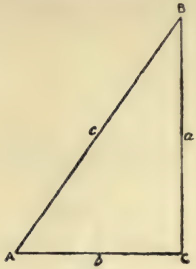

FIG. 77.

Hence $$B = \frac{\pi}{2} - A.$$

B is said to be the *complement* of A.

Now
$$\cos B = \cos\left(\frac{\pi}{2} - A\right) = \frac{a}{c} = \sin A,$$
$$\sin B = \sin\left(\frac{\pi}{2} - A\right) = \frac{b}{c} = \cos A,$$
$$\tan B = \tan\left(\frac{\pi}{2} - A\right) = \frac{b}{a} = \cot A,$$
$$\cot B = \cot\left(\frac{\pi}{2} - A\right) = \frac{a}{b} = \tan A.$$

TRIGONOMETRICAL RATIOS OF $\frac{\pi}{2} + \theta$.

Let the angle $DOA = \frac{\pi}{2}$, and let the angle $BOD = \theta$ (Fig. 78). Then the angle $BOA = \frac{\pi}{2} + \theta$.

$$\sin\left(\frac{\pi}{2} + \theta\right) = \frac{CB}{OB} = \frac{OD}{OB} = \cos BOD = \cos\theta.$$

Since θ is less than $\frac{\pi}{2}$, $\sin\theta$ and $\cos\theta$ will be positive. Hence, remembering that OC will be negative, we get—

$$\cos\left(\frac{\pi}{2} + \theta\right) = \frac{OC}{OB} = \frac{DB}{OB} = -\sin\theta,$$

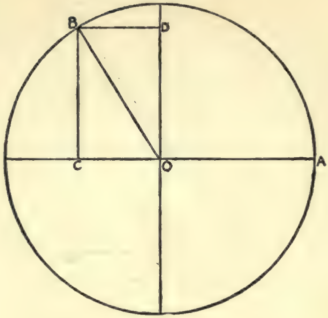

FIG. 78.

$$\tan\left(\frac{\pi}{2}+\theta\right)=\frac{\mathsf{CB}}{\mathsf{OC}}=\frac{\mathsf{OD}}{\mathsf{DB}}=-\cot\theta,$$

$$\cot\left(\frac{\pi}{2}+\theta\right)=\frac{\mathsf{OC}}{\mathsf{CB}}=\frac{\mathsf{DB}}{\mathsf{OD}}=-\tan\theta.$$

The results already deduced, from page 241 onward to the present point, may also be obtained by inspection of the curves given in Fig. 4, page 14.

To prove that $\sin^2\theta+\cos^2\theta=1$, *whatever value may be given to* θ.

In any right-angled triangle, let p = the length of the perpendicular, let b = the length of the base, and let h = the length of the hypotenuse. If the angle opposite to the perpendicular is equal to θ, we have—

$$\sin\theta=\frac{p}{h},\ \cos\theta=\frac{b}{h}.$$

$$\therefore\ \sin^2\theta+\cos^2\theta=\frac{p^2}{h^2}+\frac{b^2}{h^2}=\frac{p^2+b^2}{h^2}=\frac{h^2}{h^2}=1,$$

since $$p^2+b^2=h^2.\quad \text{(Euclid, I. 47.)}$$

Corollaries:—

1. $\sin\theta=\sqrt{1-\cos^2\theta}.$
2. $\cos\theta=\sqrt{1-\sin^2\theta}.$
3. $\tan\theta=\frac{p}{b}=\frac{\frac{p}{h}}{\frac{b}{h}}=\frac{\sin\theta}{\cos\theta}=\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}=\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}.$

4. $1+\tan^2\theta = 1+\dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{\cos^2\theta+\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}.$

$$\therefore \cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}}.$$

5. $1+\cot^2\theta = 1+\dfrac{\cos^2\theta}{\sin^2\theta} = \dfrac{\sin^2\theta+\cos^2\theta}{\sin^2\theta} = \dfrac{1}{\sin^2\theta}.$

$$\therefore \sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}}.$$

6. Since $1+\cot^2\theta = 1+\dfrac{1}{\tan^2\theta} = \dfrac{1+\tan^2\theta}{\tan^2\theta}$, we have, from the result of 5 above,

$$\sin\theta = \frac{1}{\sqrt{\dfrac{1+\tan^2\theta}{\tan^2\theta}}} = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}}.$$

To determine the value of $\sin(\theta+\phi)$ *and* $\cos(\theta+\phi)$ *in terms of* $\sin\theta$, $\cos\theta$, $\sin\phi$, *and* $\cos\phi$.

Let the angle BOA (Fig. 79) be equal to θ, and let the angle BOC be equal to ϕ. Then the angle COA will be equal to $\theta+\phi$.

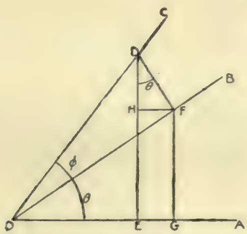

Fig. 79.

From D, any point in OC, drop the perpendiculars DE on OA, and DF on OB. From F drop the perpendiculars FG on OA, and FH on DE.

Then, since HDF is a right-angled triangle, $\angle \text{HDF} + \angle \text{DFH} = \dfrac{\pi}{2}$.

Also, $\angle \text{DFH} + \angle \text{HFO} = \dfrac{\pi}{2}$, since $\angle \text{DFO}$ is a right angle.

$$\therefore \angle \text{HDF} = \angle \text{HFO} = \angle \text{FOA} = \theta,$$

since HF is parallel to OA.

Now

$$\sin(\theta+\phi) = \sin \angle \mathsf{DOE} = \frac{\mathsf{DE}}{\mathsf{OD}} = \frac{\mathsf{DH}+\mathsf{HE}}{\mathsf{OD}} = \frac{\mathsf{DH}}{\mathsf{OD}} + \frac{\mathsf{HE}}{\mathsf{OD}} = \frac{\mathsf{DH}}{\mathsf{OD}} + \frac{\mathsf{FG}}{\mathsf{OD}}.$$

Multiply both numerator and denominator of $\frac{\mathsf{DH}}{\mathsf{OD}}$ by DF, and multiply both numerator and denominator of $\frac{\mathsf{FG}}{\mathsf{OD}}$ by FO. Then

$$\sin(\theta+\phi) = \frac{\mathsf{DH}}{\mathsf{DF}} \cdot \frac{\mathsf{DF}}{\mathsf{OD}} + \frac{\mathsf{FG}}{\mathsf{FO}} \cdot \frac{\mathsf{FO}}{\mathsf{OD}} = \cos\theta \sin\phi + \sin\theta\cos\phi.$$

Corollary.—Let $\phi=\theta$. Then $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\cos(\theta+\phi) = \cos \angle \mathsf{DOE} = \frac{\mathsf{OE}}{\mathsf{OD}} = \frac{\mathsf{OG}-\mathsf{EG}}{\mathsf{OD}} = \frac{\mathsf{OG}}{\mathsf{OD}} - \frac{\mathsf{HF}}{\mathsf{OD}}.$$

Multiply both numerator and denominator of $\frac{\mathsf{OG}}{\mathsf{OD}}$ by OF, and multiply both numerator and denominator of $\frac{\mathsf{HF}}{\mathsf{OD}}$ by DF. Then

$$\cos(\theta+\phi) = \frac{\mathsf{OG}}{\mathsf{OF}} \cdot \frac{\mathsf{OF}}{\mathsf{OD}} - \frac{\mathsf{HF}}{\mathsf{DF}} \cdot \frac{\mathsf{DF}}{\mathsf{OD}} = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

Corollary.—Let $\phi=\theta$. Then $\cos 2\theta = \cos^2\theta - \sin^2\theta$.

To determine the value of $\sin(\theta-\phi)$ *and* $\cos(\theta-\phi)$ *in terms of* $\sin\theta$, $\cos\theta$, $\sin\phi$, *and* $\cos\phi$.

Let the angle BOA (Fig. 80) be equal to θ, and let the angle BOC be equal to ϕ. Then the angle COA is equal to $(\theta-\phi)$.

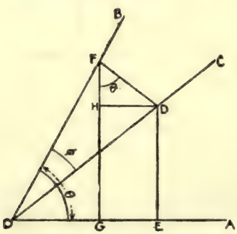

Fig. 80.

From any point D on OC drop the perpendiculars DF on OB, and DE on OA. From F drop the perpendicular FG on OA, and draw DH parallel to OA.

Then, since FOG is a right-angled triangle, $\angle FOG + \angle OFG = \frac{\pi}{2}$.

Also, $\angle OFG + \angle HFD = \frac{\pi}{2}$, since D is perpendicular to OB. Hence $\angle FOG = \angle HFD$.

$$\therefore \angle HFD = \theta.$$

$$\sin(\theta - \phi) = \sin \angle DOE = \frac{DE}{OD} = \frac{FG - HF}{OD} = \frac{FG}{OD} - \frac{HF}{OD}.$$

Multiply both numerator and denominator of $\frac{FG}{OD}$ by OF, and multiply both numerator and denominator of $\frac{HF}{OD}$ by FD. Then

$$\sin(\theta - \phi) = \frac{FG}{OF} \cdot \frac{OF}{OD} - \frac{HF}{FD} \cdot \frac{FD}{OD} = \sin\theta \cos\phi - \cos\theta \sin\phi.$$

This result may also be obtained by substituting $-\phi$ for ϕ in the expression for $\sin(\theta + \phi)$.

When $\phi = \theta$,

$$\sin(\theta - \theta) = \sin 0 = \sin\theta \cos\theta - \cos\theta \sin\theta = 0.$$

$$\cos(\theta - \phi) = \cos \angle DOE = \frac{OE}{OD} = \frac{OG + GD}{OD} = \frac{OG}{OD} + \frac{HD}{OD}.$$

Multiply both numerator and denominator of $\frac{OG}{OD}$ by OF, and multiply both numerator and denominator of $\frac{HD}{OD}$ by FD. Then

$$\cos(\theta - \phi) = \frac{OG}{OF} \cdot \frac{OF}{OD} + \frac{HD}{FD} \cdot \frac{FD}{OD} = \cos\theta \cos\phi + \sin\theta \sin\phi.$$

This result may also be obtained by substituting $-\phi$ for ϕ in the expression for $\cos(\theta + \phi)$.

When $\phi = \theta$,

$$\cos(\theta - \theta) = \cos 0 = \cos^2\theta + \sin^2\theta = 1.$$

To prove that $\cos\theta = \sqrt{\frac{1 + \cos 2\theta}{2}}$,

and that $\sin\theta = \sqrt{\frac{1 - \cos 2\theta}{2}}$.

We have

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$1 = \cos^2\theta + \sin^2\theta.$$

Adding these two equations, we get

$$1 + \cos 2\theta = 2 \cos^2\theta,$$

$$\cos \theta = \sqrt{\frac{1 + \cos 2\theta}{2}}.$$

Subtracting the upper from the lower equation, we get

$$1 - \cos 2\theta = 2 \sin^2\theta.$$

$$\therefore \sin \theta = \sqrt{\frac{1 - \cos 2\theta}{2}}.$$

To prove the formulæ

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2},$$

$$\cos B - \cos A = 2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$$

Let θ and ϕ be any angles, then

$$\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi,$$
$$\cos (\theta - \phi) = \cos \theta \cos \phi + \sin \theta \sin \phi.$$

Adding these two equations, we get

$$\cos (\theta + \phi) + \cos (\theta - \phi) = 2 \cos \theta \cos \phi. \quad . \quad (1).$$

Subtracting the upper equation from the lower, we get

$$\cos (\theta - \phi) - \cos (\theta + \phi) = 2 \sin \theta \sin \phi. \quad . \quad (2).$$

Now let $\quad A = \theta + \phi, \quad B = \theta - \phi.$

Then $\quad \theta = \frac{A + B}{2}, \quad \phi = \frac{A - B}{2}.$

Hence, substituting in (1), we get

$$\cos A + \cos B = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2}.$$

Substituting in (2), we get

$$\cos B - \cos A = 2 \sin \frac{A + B}{2} \sin \frac{A - B}{2}.$$

By writing down the formulæ for $\sin (\theta + \phi)$ and $\sin (\theta - \phi)$, and proceeding in a manner similar to that just employed, we may prove the formulæ

$$\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2},$$

$$\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}.$$

To find the area of a triangle, in terms of the product of two sides and the sine of the included angle.

Let ABC (Fig. 81) be the given triangle; it is required to

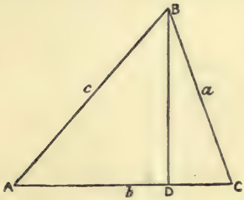

FIG. 81.

express its area in terms of the sides AB and AC, and the sine of A.

Drop the perpendicular BD from B on to AC. Then

$$\frac{BD}{c} = \sin A. \quad \therefore \; BD = c \sin A.$$

But area of triangle $ABC = AC \times \frac{1}{2} BD = \frac{1}{2} bc \sin A.$

To show that in any triangle

$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$

In Fig. 81, let h be the length of the perpendicular BD. Then

$$\frac{h}{a} = \sin C. \quad \therefore \; h = a \sin C,$$

$$\frac{h}{c} = \sin A. \quad \therefore \; h = c \sin A.$$

$$\therefore \; a \sin C = c \sin A,$$

$$\frac{\sin A}{\sin C} = \frac{a}{c}.$$

To prove that in any triangle

$$c^2 = a^2 + b^2 - 2ab \cos \mathrm{C}.$$

In Fig. 81, let $\mathrm{BD} = h$, $\mathrm{DC} = x$.

Then $\mathrm{AD} = (b - x)$.

Also, $\frac{x}{a} = \cos \mathrm{C}. \quad \therefore x = a \cos \mathrm{C}.$

By Euclid, I. 47,

$$a^2 = h^2 + x^2. \qquad \therefore h^2 = a^2 - x^2.$$

Also, $c^2 = h^2 + (b - x)^2. \quad \therefore h^2 = c^2 - (b - x)^2.$

$$\therefore c^2 - (b - x)^2 = a^2 - x^2.$$

$$\begin{aligned} \therefore c^2 &= a^2 - x^2 + (b - x)^2 \\ &= a^2 - x^2 + b^2 - 2bx + x^2 \\ &= a^2 + b^2 - 2bx \\ &= a^2 + b^2 - 2ab \cos \mathrm{C}. \end{aligned}$$

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THE END.